EXAM IN FOURIER ANALYSIS, TMA362/MMG710

CHALMERS / GÖTEBORGS UNIVERSITET

OCTOBER 27, 2018

Examinator: Michael Björklund,
Telefonvakt: Barbara Schnitzer, anknytning 5325

Help: Only the formulas on the next page!

Grades:
TMA362: 3 = 12-16 points, 4 = 17-21, 5 = 22-25.
MMG710: G = 12-17 points, VG = 18-25 points.

Problem 1. (4+2 points)

a) Show that if $f : \mathbb{R} \to \mathbb{C}$ is a continuously differentiable 1-periodic function, then its Fourier coefficients are absolutely summable, i.e. $\sum_{m=-\infty}^{\infty} |c_f(m)| < \infty$.

b) Show that if $f : \mathbb{R} \to \mathbb{C}$ is a continuous 1-periodic function, then

$$\sum_{m=-\infty}^{\infty} |c_f(m)|^4 \leq \Big(\int_0^1 |f(x)|^2 \, dx \Big)^2.$$

Note: You are free to use Bessel's and Cauchy-Schwarz'-inequalities without proofs in both a) and b)

Problem 2 (5 points)**.** Find a differentiable function $f : (0, \infty) \to \mathbb{R}$ such that

$$\int_0^t f'(u) f(t-1-u) \, du = \begin{cases} 0 & \text{for } 0 < t < 1 \\ (t-1)e^t & \text{for } t \geq 1 \end{cases}$$

Hint: It may or may not be useful to know that for every integer $n \geq 0$:

$$\int_0^\infty t^n e^{-at} \, dt = \frac{n!}{a^{n+1}}, \quad \text{for all } a \in \mathbb{C} \text{ with } \mathrm{Re}(a) > 0.$$

Problem 3 (5 points)**.** Show that for all $\alpha, \beta > 0$, there is an *explicit* constant $\gamma(\alpha, \beta) > 0$ such that

$$\int_{-\infty}^\infty \frac{1}{(\alpha^2 + y^2)(\beta^2 + (x-y)^2)} \, dy = \frac{\gamma(\alpha, \beta)}{(\alpha+\beta)^2 + x^2}, \quad \text{for all } x \in \mathbb{R}.$$

For full points, a formula for $\gamma(\alpha, \beta)$ has to be given. *Hint: You are free to use (without proof) the identity:*

$$\int_0^\infty \frac{\cos(tx)}{1+x^2} \, dx = \frac{\pi}{2} e^{-t} \quad \textit{for all } t \geq 0.$$

Problem 4 (5+4 points). a) Find an explicit sequence (b_k) of real numbers such that

$$x^2 \sin 2\pi x = \sum_{k=1}^{\infty} b_k \sin(2\pi k x), \quad \text{for all } 0 \leq x \leq 1/2.$$

It may or may not be useful to know that (without proofs):

$$\sin\alpha \sin\beta = \frac{1}{2}\big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\big), \quad \text{for all } \alpha, \beta \in \mathbb{R},$$

and

$$\int_0^{1/2} x^2 \cos(2\pi l x)\, dx = \frac{(-1)^l}{2\pi^2 l^2}, \quad \text{for all } l \in \mathbb{Z} \setminus \{0\}.$$

b) Find a function $u : [0,\infty) \times [0, 1/2] \to \mathbb{R}$ such that

(i) $u_t'(t,x) = u_{xx}''(t,x)$ for all $t > 0$ and $0 < x < 1/2$.

(ii) $u(t,0) = u(t,1/2) = 0$, for all $t > 0$.

(iii) $u(0,x) = x^2 \sin(2\pi x)$, for all $0 \leq x \leq 1/2$.

If you have not been able to solve a), express your answer to b) in terms of (b_k).

FORMULAS TO USE WITHOUT PROOFS

Fourier series: If f is 1-periodic, then $c_f(m) := \int_0^1 f(x) e^{-2\pi i m x}\, dx$ for $m \in \mathbb{Z}$, and

$$c_{f'}(m) = 2\pi i m c_f(m) \quad \text{for all } m \in \mathbb{Z}.$$

Fourier transform: If $f : \mathbb{R} \to \mathbb{C}$ is absolutely Riemann-integrable, then

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx, \quad \text{for } \xi \in \mathbb{R}.$$

If $f, g : \mathbb{R} \to \mathbb{C}$ are absolutely Riemann integrable, then so is their convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x-y)\, dy,$$

and $\widehat{f * g} = \widehat{f} * \widehat{g}$.

Laplace transform: If $f : \mathbb{R} \to \mathbb{C}$ is right sided and exponentially integrable (rsei), then

$$F(s) = \mathcal{L}(f)(s) = \int_0^{\infty} f(t) e^{-st}\, dt, \quad \text{for } \mathrm{Re}(s) \text{ large enough.}$$

and if $f(t) = g(t - t_o)$ for some rsei g and $t_o \geq 0$, then $F(s) = e^{-st_o} G(s)$, and if $f = g'$, then $F(s) = sG(s) - g(0)$. Finally, if f and g are rsei, then so is $f * g$ and $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$.