EXAM IN FOURIER ANALYSIS, TMA362/MMG710

CHALMERS / GÖTEBORGS UNIVERSITET

JANUARY 9, 2019

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Help: Only the formulas on the next page!

Grades:
TMA362: 3 = 12-16 points, 4 = 17-21, 5 = 22-25.
MMG710: G = 12-17 points, VG = 18-25 points.

Problem 1. (5+5 points)

a) Show that if $f : \mathbb{R} \to \mathbb{C}$ is continuous and satisfies

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |\widehat{f}(\xi)|\, d\xi < \infty,$$

then

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i \xi x}\, d\xi, \quad \text{for all } x \in \mathbb{R}.$$

Hint: You are free to use Fejer's Theorem for Fourier transforms without proof.

b) Show that

$$\frac{\pi}{2} e^{-|t|} = \int_0^{\infty} \frac{1}{1+u^2} \cos tu\, du, \quad \text{for all } t \in \mathbb{R}.$$

Note: You are free to use the result in a) even if you have not been able to prove it.

Problem 2 (4 points)**.** Show that if $f(x) = (ax+b)e^{-\pi x^2}$, where $a, b \in \mathbb{C}$, then

$$\widehat{f}(\xi) = (c\xi + d)e^{-\pi \xi^2},$$

for some $c, d \in \mathbb{C}$ which depend linearly on a and b. Determine a complex 2×2 matrix T such that

$$\begin{pmatrix} c \\ d \end{pmatrix} = T \begin{pmatrix} a \\ b \end{pmatrix}, \quad \text{for all } a, b \in \mathbb{C},$$

and calculate its eigenvalues.
Hint: You are free to use without proof that $g(x) = e^{-\pi x^2}$ has Fourier transform $\widehat{g}(\xi) = e^{-\pi \xi^2}$.

Problem 3 (3 points)**.** Calculate the Laplace transform of the function $f : [0, \infty) \to \mathbb{C}$ given by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ (1-x)e^{-x} & \text{if } x \ge 1. \end{cases}$$

Hint: You are free to use that:

$$\int_0^\infty te^{-st}\,dt = \frac{1}{s^2}, \quad \textit{for all } \mathrm{Re}(s) > 0.$$

Problem 4 (4+4 points)**.** a) Find an explicit sequence (b_k) of real numbers such that

$$x(1-2x) = \sum_{k=1}^{\infty} b_k \sin(2\pi kx), \quad \text{for all } 0 \le x \le 1/2.$$

It may or may not be useful to know that (without proofs):

$$\int_0^{1/2} x \sin 2\pi nx\,dx = \frac{(-1)^{n+1}}{4\pi n}$$

and

$$\int_0^{1/2} x^2 \sin 2\pi nx\,dx = \frac{(-1)^{n+1}}{8\pi n} - \frac{(1-(-1)^n)}{4\pi n^3},$$

for $n \ge 1$.

b) Find a function $u : [0, \infty) \times [0, 1/2] \to \mathbb{R}$ such that

(i) $u_t'(t,x) = u_{xx}''(t,x)$ for all $t > 0$ and $0 < x < 1/2$.

(ii) $u(t,0) = u(t,1/2) = 0$, for all $t > 0$.

(iii) $u(0,x) = x(1-2x)$, for all $0 \le x \le 1/2$.

If you have not been able to solve a), express your answer to b) in terms of (b_k).

Formulas to use without proofs

Fourier series: If f is 1-periodic, then $c_f(m) := \int_0^1 f(x)e^{-2\pi imx}\,dx$ for $m \in \mathbb{Z}$.

Fourier transform: If $f : \mathbb{R} \to \mathbb{C}$ is absolutely Riemann-integrable, then its Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}\,dx, \quad \text{for } \xi \in \mathbb{R}.$$

If f, f' are absolutely Riemann integrable, then $\widehat{f'}(\xi) = 2\pi i\xi\widehat{f}(\xi)$ for all $\xi \in \mathbb{R}$.

Laplace transform: If $f : \mathbb{R} \to \mathbb{C}$ is right sided and exponentially integrable (rsei), then

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt, \quad \text{for } \mathrm{Re}(s) \text{ large enough.}$$