Solutions to the exam in TMA362 / MMG710

[October 27, 2018]

1a) $c_{f'}(m) = 2\pi i m \cdot c_f(m) \Rightarrow \forall m \neq 0 \quad |c_f(m)| \leqslant \frac{|c_{f'}(m)|}{2\pi |m|}$

$$\rightsquigarrow \sum_{m=-\infty}^{\infty} |c_f(m)| = \underbrace{\left| \int_0^1 f(x)\,dx \right|}_{<+\infty} + \underbrace{\sum_{m\neq 0} \frac{1}{2\pi |m|} \cdot |c_{f'}(m)|}$$

$$(\text{Cauchy-Schwarz}) \leqslant \underbrace{\frac{1}{2\pi} \cdot \underbrace{\left(\sum_{m \neq 0} \frac{1}{m^2} \right)^{1/2}}_{<+\infty} \underbrace{\left(\sum_{m\neq 0} |c_{f'}(m)|^2 \right)^{1/2}}_{(\text{Bessel}) \leqslant \int_0^1 |f'(x)|^2 dx < +\infty}}_{<+\infty}.$$

b) Cauchy-Schwarz: $|c_f(m)| \leqslant \int_0^1 |f(x)|\,dx \leqslant \left(\int_0^1 |f(x)|^2 dx \right)^{1/2} \quad \forall m \in \mathbb{Z}$

$$\sum_{m=-\infty}^{\infty} |c_f(m)|^4 \leqslant \Big(\underbrace{\sum_{m=-\infty}^{\infty} |c_f(m)|^2}_{(\text{Bessel}) \leqslant \int_0^1 |f(x)|^2 dx} \Big) \cdot \int_0^1 |f(x)|^2 dx \Big) \leqslant \left(\int_0^1 |f(x)|^2 dx \right)^2$$

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2. $g(t) = f(t-1)$, f right-sided $\rightsquigarrow G(s) = F(s) e^{-s}$

$$h(t) = \begin{cases} e \cdot t e^{t} & t \geqslant 0 \\ 0 & t < 0 \end{cases} \quad \rightsquigarrow \quad H(s) = \frac{e}{(s-1)^2}$$

Want to solve: $(f' * g)(t) = h(t-1)$

Laplace

$$\rightsquigarrow \quad (sF(s) - f(0))\, G(s) = H(s) e^{-s}$$

$$\Longleftrightarrow \quad (sF(s) - f(0))\, F(s) = \frac{e}{(s-1)^2}$$

Set: $F(s) = \frac{\sqrt{e}}{s-1}$ so that $f(t) = \sqrt{e} \cdot e^{t} \quad t \geqslant 0 \quad (f(0) = \sqrt{e})$

$= 0 \quad t < 0$

$$e \cdot \left(\frac{s}{s-1} - 1 \right) \frac{1}{s-1} = \frac{e}{(s-1)^2}$$

$$\rightsquigarrow \boxed{f(t) = \begin{cases} \sqrt{e} \cdot e^{t} & t \geqslant 0 \\ 0 & t < 0. \end{cases}}$$

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3. Set $f_\alpha(x) = \frac{1}{\alpha^2 + x^2}$ so that $f_\alpha * f_\beta = \gamma(\alpha,\beta)\, f_{\alpha+\beta}$ want to find!

$\left(\Longleftrightarrow \quad \hat{f}_\alpha \cdot \hat{f}_\beta = \gamma(\alpha,\beta)\, \hat{f}_{\alpha+\beta} \right)$

<u>Need</u>: $\hat{f}_\alpha(\xi) = \int_{-\infty}^{\infty} \frac{1}{\alpha^2 + x^2} e^{-2\pi i \xi x}\, dx = 2 \int_0^\infty \frac{\cos 2\pi \xi x}{\alpha^2 + x^2}\, dx$

$= \frac{2}{\alpha} \int_0^\infty \frac{\cos\left(2\pi\xi\alpha \cdot \frac{x}{\alpha}\right)}{1 + \left(\frac{x}{\alpha}\right)^2} \frac{dx}{\alpha} = \frac{2}{\alpha} \int_0^\infty \frac{\cos\left(\overset{= t}{2\pi\xi\alpha}\, y\right)}{1 + y^2}\, dy$

$= \frac{\cancel{2}}{\alpha} \cdot \frac{\pi}{\cancel{2}} e^{-2\pi\xi\alpha}, \quad \xi \geq 0.$

$\rightsquigarrow \quad \hat{f}_\alpha(\xi)\, \hat{f}_\beta(\xi) = \frac{\pi}{\alpha\beta} e^{-2\pi\xi(\alpha+\beta)} = \overset{\gamma(\alpha,\beta)}{\frac{\alpha+\beta}{\alpha\beta}} \underbrace{\frac{\pi}{\alpha+\beta} e^{-2\pi\xi(\alpha+\beta)}}_{= \hat{f}_{\alpha+\beta}(\xi)}$

4.a) $f(x) = x^2 \sin 2\pi x, \quad -\frac{1}{2} \leq x < \frac{1}{2}$ (f odd + real-valued)

$c_f(m) = \int_{-1/2}^{1/2} x^2 \sin 2\pi x \cdot e^{-2\pi i m x}\, dx = -i \int_{-1/2}^{1/2} x^2 \sin 2\pi x \cdot \sin 2\pi m x\, dx$

$= -2i \underbrace{\int_0^{1/2} x^2 \sin 2\pi x \cdot \sin 2\pi m x\, dx}_{= \Theta_m}$. <u>Note</u>: $c_f(-m) = -c_f(m) \;\forall m$, $c_f(0) = 0$

$\rightsquigarrow \quad f(x) = \sum_{m=-\infty}^{\infty} c_f(m) e^{2\pi i m x} = \sum_{m=1}^{\infty} \left(c_f(m) e^{2\pi i m x} + c_f(-m) e^{-2\pi i m x} \right)$

$= \sum_{m=1}^{\infty} \underbrace{2i\, c_f(m)}_{\substack{= 2i(-2i)\Theta_m \\ = 4 \cdot \Theta_m}} \cdot \underbrace{\left(\frac{e^{2\pi i m x} - e^{-2\pi i m x}}{2i} \right)}_{\sin 2\pi m x} = 4 \sum_{m=1}^{\infty} \Theta_m \sin 2\pi m x$

$\rightsquigarrow \quad \beta_k = 4\Theta_k = 4 \cdot \int_0^{1/2} x^2 \sin 2\pi x \sin 2\pi k x\, dx = \{\text{formula sheet}\}$

$= 2 \left(\int_0^{1/2} x^2 \sin 2\pi (1-k) x\, dx - \int_0^{1/2} x^2 \sin 2\pi (1+k) x\, dx \right)$

$= \{\text{formula sheet}, k \neq 1\} = 2 \left(\frac{(-1)^{1-k}}{2\pi^2 (1-k)^2} - \frac{(-1)^{1+k}}{2\pi^2 (1+k)^2} \right)$

$= -\frac{4(-1)^k}{\pi^2} \left(\frac{k}{(1-k)^2 \cdot (1+k)^2} \right), \quad k \neq 1$

$\beta_1 = -\frac{2 \cdot (-1)^{1+1}}{2\pi^2 \cdot 4} = -\frac{1}{4\pi^2}.$

b) Ansatz: $u(x,t) = X(x)T(t) \rightsquigarrow T' \cdot X = T X'' \rightsquigarrow \frac{T'}{T} = \frac{X''}{X} = -\lambda^2$

$$\rightsquigarrow \begin{cases} T(t) = C_0 e^{-\lambda^2 t} \\ X(x) = A_0 \cos \lambda x + B_0 \sin \lambda x \end{cases}$$

$$\rightsquigarrow \quad u(x,t) = e^{-\lambda^2 t} \left(A \cos \lambda x + B \sin \lambda x \right)$$

$u(0,t) = 0 \ \forall t \rightsquigarrow A = 0$

$u(\frac{1}{2}, t) = 0 \ \forall t \rightsquigarrow \sin \frac{\lambda}{2} = 0 \rightsquigarrow \lambda = 2\pi k$, k heltal (wlog, $k \geq 1$)

$\rightsquigarrow u_k(x,t) = \beta_k e^{-4\pi^2 k^2 t} \sin 2\pi k x$ solves i)+ii) $\forall k \geq 1$

$\rightsquigarrow u(x,t) = \sum_{k=1}^{\infty} \beta_k e^{-4\pi^2 k^2 t} \sin 2\pi k x$ — " —

$$u(x,0) = \sum_{k=1}^{\infty} \beta_k \sin 2\pi k x \overset{a)}{\rightsquigarrow} \beta_k = \begin{cases} -\frac{1}{4\pi^2} & k = 1 \\ -\frac{4(-1)^k}{\pi^2} \frac{k}{(k-1)^2(k+1)^2} & k \neq \pm 1 \end{cases}$$

$$\rightsquigarrow \boxed{u(x,t) = -\frac{e^{-4\pi^2 t}}{4\pi^2} \sin 2\pi x - \frac{4}{\pi^2} \sum_{k=2}^{\infty} \frac{(-1)^k k}{(k^2-1)^2} \sin 2\pi k x \cdot e^{-4\pi^2 k^2 t}}$$