(FS) Bonus points for the exam in MMG710

$N \geq 1$ integer, $\mathbb{Z}/N\mathbb{Z} = \{0, 1, \dots, N-1\}$ with addition modulo N.

Ex. $N = 5$, $3 + 4 = 7 \bmod 5 = 2$, $2 + 3 = 5 \pmod 5 = 0$.

$\ell^2(\mathbb{Z}/N\mathbb{Z}) = \{ f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C} \}$ (cplx. vector space: $(\lambda \cdot f)(x) = \lambda f(x)$, $(f_1 + f_2)(x) = f_1(x) + f_2(x)$)

$\langle f_1, f_2 \rangle = \frac{1}{N} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} f_1(x) \overline{f_2(x)}$, $f_1, f_2 \in \ell^2(\mathbb{Z}/N\mathbb{Z})$

1) Show that $(\ell^2(\mathbb{Z}/N\mathbb{Z}), \langle \cdot, \cdot \rangle)$ is a Hilbert space of dimension N. (1p)

2) For $m \in \mathbb{Z}/N\mathbb{Z}$, set $e_m(x) := e^{\frac{2\pi i m x}{N}}$, $x \in \mathbb{Z}/N\mathbb{Z}$

Show that e_m is well-defined, and that $\{e_0, e_1, \dots, e_{N-1}\}$ is an orthonormal basis for $(\ell^2(\mathbb{Z}/N\mathbb{Z}), \langle \cdot, \cdot \rangle)$. (2p)

3) [Parseval's formula] If $f \in \ell^2(\mathbb{Z}/N\mathbb{Z})$, set

$$\hat{f}(m) := \langle f, e_m \rangle = \frac{1}{N} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} f(x) e^{-\frac{2\pi i m x}{N}}, \quad \text{for } m \in \mathbb{Z}/N\mathbb{Z}$$

Show that $\|\hat{f}\|_{\ell^2(\mathbb{Z}/N\mathbb{Z})} = c_N \cdot \|f\|_{\ell^2(\mathbb{Z}/N\mathbb{Z})}$ for some $c_N > 0$, (explicit!) and for all $f \in \ell^2(\mathbb{Z}/N\mathbb{Z})$. (3p)

4) If $f_1, f_2 : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, set $(f_1 * f_2)(x) = \frac{1}{N} \sum_{y \in \mathbb{Z}/N\mathbb{Z}} f_1(y) f_2(x-y)$, for $x \in \mathbb{Z}/N\mathbb{Z}$ (convolution). If $N = 2M$, find

$$k_M : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C} \text{ s.t. } \widehat{(k_M * f)}(m) = \begin{cases} \hat{f}(m) & m = 0, \dots, M-1 \\ 0 & m = M, \dots, N-1 \end{cases} \quad \forall f \in \ell^2(\mathbb{Z}/N\mathbb{Z})$$

(4p)

5) Can you find $f \in \ell^2(\mathbb{Z}/7\mathbb{Z})$ $(f \neq 0)$ s.t. $\hat{f}(m) = f(m)$ $\forall m \in \mathbb{Z}/7\mathbb{Z}$? (2p)

Total of 12 p

Good luck ☺

1 Bonus point: 6 – 9 pts.

2 Bonus points: 10 – 12 pts

$\boxed{\text{Hand in before: Sep 20, at 12.00.}}$!