MODULE II

- Isoperimetric inequality and Wirtinger's inequality.
- Bessel's inequality and Parseval's formula.
- Bernstein's Lemma.

Aim: Give an example of how Parseval's formula can be used in variational problems (Isoperimetric inequality), as well as in establishing convergence of Fourier series (Bernstein's Lemma).

Recommended exercises:

1, 3, 4*, 5, 6, 7, 10

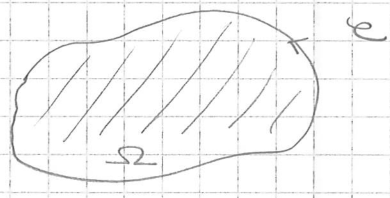
Important to focus on:

- Cauchy-Schwarz inequality
- Bessel's inequality
- Parseval's formula.

Isoperimetric inequality

Problem: Given $L > 0$, find a (differentiable) closed curve $\mathcal{C} \subset \mathbb{R}^2$ of length L, which maximizes the enclosed area.

Answer: Circle of radius $\frac{L}{2\pi}$.



How to prove this?

Let $\mathcal{C} \subset \mathbb{R}^2 \cong \mathbb{C}$ be a differentiable closed curve and fix a differentiable parameterization $\gamma : [0,1] \to \mathbb{C}$, $\gamma(0) = \gamma(1)$, i.e. $\mathcal{C} = \gamma([0,1]) \subset \mathbb{C}$.

Recall: $L = \text{Length}(\mathcal{C}) = \int_0^1 |\gamma'(t)|\, dt$ (independent of parameterization).

It will be convenient to pass to the arc-length parameterization, i.e. let $\varphi : [0,1] \circlearrowleft$ be a re-parameterization s.t. $f = \gamma \circ \varphi$ satisfies $|f'(t)| = L \quad \forall t \in [0,1]$. (WLOG: $\int_0^1 f(t)\, dt = 0$ which can always be arranged by translating $\mathcal{C}$)

In particular: $\boxed{L^2 = \int_0^1 |f'(t)|^2\, dt}$

Green's formula: $\boxed{A = \text{Area inside } \mathcal{C} = \frac{1}{2} \operatorname{Im} \int_0^1 f'(t) \overline{f(t)}\, dt}$ (Independent of parameteriz.)

(Rmk: $A = \iint_\Omega 1\, dx\,dy = \{\text{Green}\} = \frac{1}{2} \int_{\mathcal{C}} (-y, x) \cdot d\vec{r} = \{f = u + iv\}$

$= \frac{1}{2} \int_0^1 \underbrace{(-v \cdot u' + u \cdot v')}_{= \operatorname{Im} f' \cdot \bar{f}}\, dt = \frac{1}{2} \operatorname{Im} \int_0^1 f'(t) \overline{f(t)}\, dt$)

Theorem (Isoperimetric inequality) $4\pi A \leq L^2$, and equality occurs iff $\mathcal{C}$ circle of radius $\frac{L}{2\pi}$.

The proof consists of combining two inequalities:

Lemma 1 (Wirtinger's inequality) $\forall f \in \mathcal{D}_1$ with $\int_0^1 f(t)\, dt = 0$ we have $\int_0^1 |f'(t)|^2\, dt \geq 4\pi^2 \int_0^1 |f(t)|^2\, dt$, and equality occurs iff $f(t) = A e^{2\pi i t} + B e^{-2\pi i t}$ for some $A, B \in \mathbb{C}$.

Lemma 2 (Cauchy-Schwarz) $\forall f, g \in \mathcal{D}_1$ $\left| \int f(t) \overline{g(t)}\, dt \right| \leq \left(\int |f(t)|^2\, dt \right)^{1/2} \left(\int_0^1 |g(t)|^2\, dt \right)^{1/2}$, and equality occurs iff $f = \lambda g$ for some $\lambda \in \mathbb{C}$.

Proof of the isoperimetric inequality assuming Lemma 1+2:

$$4\pi A = 2\pi \cdot \mathrm{Im} \int_0^1 f'(t)\overline{f(t)}\,dt \underset{\text{(Lemma 2)}}{\leq} 2\pi \left(\int_0^1 |f'(t)|^2 dt\right)^{1/2} \left(\int_0^1 |f(t)|^2 dt\right)^{1/2}$$

$$\underset{\text{(Lemma 1)}}{\leq} 2\pi \left(\int_0^1 |f'(t)|^2 dt\right)^{1/2} \cdot \frac{1}{2\pi} \left(\int_0^1 |f'(t)|^2 dt\right)^{1/2} = \int_0^1 |f'(t)|^2 dt = L^2$$

If equality holds, then: i) $f'(t) = \lambda \cdot f(t)$ for some $\lambda \in \mathbb{C}$ (Lemma 2)

ii) $f(t) = A e^{2\pi i t} + B e^{-2\pi i t}$ for some $A, B \in \mathbb{C}$ (Lemma 1)

$$\rightsquigarrow \quad f'(t) = 2\pi i A e^{2\pi i t} - 2\pi i B e^{-2\pi i t} = \lambda \left(A e^{2\pi i t} + B e^{-2\pi i t}\right)$$

$$\rightsquigarrow \begin{cases} A = 0 & B \neq 0 & \lambda = -2\pi i \\ A \neq 0 & B = 0 & \lambda = 2\pi i \end{cases} \quad \rightsquigarrow \quad f(t) = C e^{\pm 2\pi i t}$$

Since $|f'(t)| = |\pm 2\pi i C e^{\pm 2\pi i t}| = 2\pi |C| = L \rightsquigarrow C = \frac{L}{2\pi} e^{2\pi i \phi}$ for some $\phi \in \mathbb{R}$

$$\rightsquigarrow \quad f(t) = \frac{L}{2\pi} e^{2\pi i(\phi \pm t)}$$ (parameterize a circle of radius $L/2\pi$)

Proof of Wirtinger's inequality (Lemma 1):

$$\boxed{\text{Theorem (Parseval's formula)} \quad \forall f \in \mathcal{P}_1 \quad \sum_m |c_f(m)|^2 = \int_0^1 |f(t)|^2 dt}$$

Recall (Module 1): $c_{f'}(m) = 2\pi i m \cdot c_f(m) \quad \forall m \in \mathbb{Z}$

Parseval:
- $\int_0^1 |f'(t)|^2 dt = \sum_m |c_{f'}(m)|^2 = 4\pi^2 \sum_{m \neq 0} m^2 \cdot |c_f(m)|^2$
- $\int_0^1 |f(t)|^2 dt = \sum_m |c_f(m)|^2 = \sum_{m \neq 0} |c_f(m)|^2$ if $\int_0^1 f(t)\,dt = 0$

$$\rightsquigarrow \int_0^1 |f'(t)|^2 dt - 4\pi^2 \int_0^1 |f(t)|^2 = \sum_{m \neq 0} (m^2 - 1)\,|c_f(m)|^2 \geq 0$$

Equality $\iff c_f(m) = 0 \;\; \forall |m| \geq 2 \iff f(t) = \underset{=B}{c_f(-1)}\, e^{-2\pi i t} + \underset{=A}{c_f(1)}\, e^{2\pi i t}$

Remains to prove:
- Cauchy-Schwarz inequality (Lemma 1)
- Parseval's formula

Reductions in the proof of Parseval's formula

If $f \in \mathcal{P}_1$, set $f^*(t) = \overline{f(-t)}$.

Obs: $c_{f^*}(m) = \int_0^1 \overline{f(-t)}\, e^{-2\pi i m t}\, dt = \overline{\int_0^1 f(-t)\, e^{2\pi i m t}\, dt}$

$= \overline{\int_0^1 f(t)\, e^{-2\pi i m t}\, dt} = \overline{c_f(m)} \quad \forall m \in \mathbb{Z}.$

Recall (Module 1): If $f, g \in \mathcal{P}_1$, then $c_{f*g}(m) = c_f(m)\, c_g(m) \quad \forall m \in \mathbb{Z}$

In particular: $c_{f^* * f}(m) = c_{f^*}(m)\, c_f(m) = |c_f(m)|^2$

Lemma 3 (Bessel's inequality for Fourier series)

$\forall f \in \mathcal{P}_1 \quad \sum_{m \in \mathbb{Z}} |c_f(m)|^2 \leq \int_0^1 |f(t)|^2\, dt$

Proof of Parseval's thm assuming Bessel's inequality:

Since $\int_0^1 |f(t)|^2 dt < +\infty \quad \forall f \in \mathcal{P}_1$, Bessel's inequality

shows that $\sum_m |c_f(m)|^2 < +\infty$.

$\rightsquigarrow (f^* * f)(x) = \sum_m \underbrace{|c_f(m)|^2}_{= c_{f^* * f}(m)} e^{2\pi i m x}$ (by Fejér's Thm) $\forall x \in [0,1]$

$\rightsquigarrow (f^* * f)(0) = \int_0^1 |f(t)|^2\, dt = \sum_m |c_f(m)|^2$ $\quad \blacksquare$

Both Cauchy-Schwarz and Bessel's inequalities are more conveniently proved in the context of pre-Hilbert spaces.

Note that $\mathcal{P}_1$ is an (infinite-dimensional) complex vector space. The form

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\, dt, \quad f, g \in \mathcal{P}_1 \qquad (*)$$

is well-defined, linear in the first variable (f), and skew-linear in the second; also $\langle f, f \rangle \geq 0 \quad \forall f \in \mathcal{P}_1$ with equality iff $f \equiv 0$.

This is a special case of a pre-Hilbert space.

Pre-Hilbert spaces

Def. Let V be a (possibly infinite dimensional) complex vector space. An inner product on V is a map $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{C}$ satisfying:

i) $\langle \alpha u, v \rangle = \alpha \langle u,v \rangle$ and $\langle u+v, w \rangle = \langle u,w \rangle + \langle v,w \rangle$

for all $u,v,w \in V$ and $\alpha \in \mathbb{C}$ ($\langle \cdot,\cdot \rangle$ is linear in first vaiable)

ii) $\overline{\langle u,v \rangle} = \langle v,u \rangle \quad \forall u,v \in V$

iii) $\|u\|^2 := \langle u,u \rangle \geq 0 \quad \forall u \in V$

We say that $\langle \cdot,\cdot \rangle$ is non-degenerate if $\|u\| = 0 \Rightarrow u = 0$.

A pair $(V, \langle \cdot,\cdot \rangle)$ with $\langle \cdot,\cdot \rangle$ non-degenerate is called a pre-Hilbert space.

Ex. • $(\mathcal{P}_1, \langle \cdot,\cdot \rangle)$ with $\langle f,g \rangle = \int_0^1 f(t)\overline{g(t)}\,dt$ is pre-Hilbert.

• More generally, if $w : [a,b] \to \mathbb{R}$ Riemann-integrable and $w > 0$, then $(C([a,b]), \langle \cdot,\cdot \rangle_w)$, where $\langle f,g \rangle_w = \int_a^b f(t)\overline{g(t)}\,w(t)\,dt$, is pre-Hilbert.

• If $V = \{\text{Riemann-integrable fcns on } [0,1]\}$ and $\langle f,g \rangle = \int_0^1 f(t)\overline{g(t)}\,dt$, then $(V, \langle \cdot,\cdot \rangle)$ is not pre-Hilbert. Why?

Def. $(V, \langle \cdot,\cdot \rangle)$ inner product space.

• $u,v \in V \setminus \{0\}$ orthogonal if $\langle u,v \rangle = 0$

orthonormal if $\|u\| = \|v\| = 1$ and $\langle u,v \rangle = 0$

• $\{v_1, v_2, \dots\} \subset V$ orthonormal set if $\|v_i\| = 1 \ \forall i$ and $\langle v_i, v_j \rangle = 0 \ \forall i \neq j$

Ex. $(\mathcal{P}_1, \langle \cdot,\cdot \rangle)$ above; $e_m(x) = e^{2\pi i m x}$, $m \in \mathbb{Z}$, form an orthonormal set.

Note: $c_f(m) = \langle f, e_m \rangle \quad \forall m \in \mathbb{Z}$.

In what follows, let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space.

<u>Lemma 1</u> (Cauchy-Schwarz) $\forall u, v \in V \quad |\langle u,v\rangle| \leq \|u\| \cdot \|v\| \qquad (*)$

If $\langle\cdot,\cdot\rangle$ is non-degenerate, and equality holds in $(*)$,

then $u = \lambda v$ for some $\lambda \in \mathbb{C}$.

<u>Lemma 2</u> (Bessel's inequality) If $\{v_1, v_2, \dots\} \subset V$ is an orthonormal set, then: $\forall u \in V \quad \sum_m |\langle u, v_m\rangle|^2 \leq \|u\|^2. \qquad (**)$

Before we turn to the proofs, let us collect some useful <u>formulas</u>:

1) $\|u+v\|^2 = \langle u+v, u+v\rangle \overset{i)}{=} \langle u, u+v\rangle + \langle v, u+v\rangle$

$\overset{ii)}{=} \overline{\langle u+v, u\rangle} + \overline{\langle u+v, v\rangle} \overset{i)}{=} \overline{\langle u,u\rangle} + \overline{\langle v,u\rangle} + \overline{\langle u,v\rangle} + \overline{\langle v,v\rangle}$

$\overset{iii)}{=} \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\,\langle u,v\rangle$.

2) If $\|u\| \neq 0$, set $\hat{u} = \frac{u}{\|u\|}$. By i), $\|\hat{u}\| = 1$.

Note that $(*)$ is trivial if either $\|u\| = 0$ or $\|v\| = 0$,

so to prove $(*)$ in general it suffices to show $|\langle \hat{u}, \hat{v}\rangle| \leq 1 \quad \forall \|u\|, \|v\| \neq 0$.

<u>Proof of Lemma 1</u>: By 2), it suffices to show $|\langle u,v\rangle| \leq 1$

for all $\|u\| = \|v\| = 1$ (WLOG: $\langle u,v\rangle \neq 0$)

Let $\eta \in \mathbb{C}$, with $|\eta| = 1$; By 1): For all $\|u\| = \|v\| = 1$

$\|\eta u - v\|^2 = 2\,(1 - \mathrm{Re}\,\eta\langle u,v\rangle) \geq 0 \quad$ (by iii)).

Set $\eta = \frac{\overline{\langle u,v\rangle}}{|\langle u,v\rangle|}$ so that $\|\eta u - v\|^2 = 2\,(1 - |\langle u,v\rangle|) \geq 0$

and thus: $|\langle u,v\rangle| \leq 1$ — Equality holds iff $\|\eta u - v\| = 0$.

If $\langle\cdot,\cdot\rangle$ non-degenerate $\rightsquigarrow \eta u = v$.

<u>Cor.</u> $\forall f, g \in C([a,b])$ $\quad \left| \int f(t) g(t) w(t)\, dt \right| \leq \left(\int |f(t)|^2 w(t)\, dt \right)^{1/2}$
$\forall w : [a,b] \to (0,\infty)$
Riemann-integrable $\quad \cdot \left(\int |g(t)|^2 w(t)\, dt \right)^{1/2}$

Equality occurs $\iff f(t) = \lambda g(t) \quad \forall t \in [a,b]$
for some $\lambda \in \mathbb{C}$.

Proves Lemma 2 on Page 1.

Proof of Lemma 2: Let $\{v_1, v_2, \dots\} \subset V$ be an orthonormal set.

To prove $(**)$ it suffices to prove:

$$\sum_{m=1}^{N} |\langle u, v_m\rangle|^2 \le \|u\|^2 \quad \forall u \in V,\ \forall N \geq 1. \qquad (**)_N$$

Note that $\forall N \geq 1$:

$$\|u - \sum_{m=1}^{N} \langle u, v_m\rangle v_m\|^2 \overset{1)}{=} \|u\|^2 + \underbrace{\|\sum_{m=1}^{N} \langle u, v_m\rangle v_m\|^2}_{= \sum_{m=1}^{N} |\langle u, v_m\rangle|^2 \text{ by orthonormality}} - 2\,\mathrm{Re}\, \underbrace{\langle u, \sum_{m=1}^{N} \langle u, v_m\rangle v_m\rangle}_{= \sum_{m=1}^{N} \cancel{\mathrm{Re}} |\langle u, v_m\rangle|^2}$$

$$= \|u\|^2 - \sum_{m=1}^{N} |\langle u, v_m\rangle|^2 \underset{\text{by iii)}}{\geq} 0$$

$\rightsquigarrow$ Proves $(**)_N$.

Cor. $e_m(x) = e^{2\pi i m x}$, $m \in \mathbb{Z}$, orthonormal set in $(\mathcal{P}_1, \langle \cdot, \cdot \rangle)$

and $c_f(m) = \langle f, e_m \rangle \quad \forall m \in \mathbb{Z}\ \forall f \in \mathcal{P}_1$

Lemma 2 $\rightsquigarrow$ $\displaystyle \sum_m |c_f(m)|^2 \le \int_0^1 |f(t)|^2\,dt = \|f\|^2 \quad \forall f \in \mathcal{P}_1$

Proves Lemma 3 on Page 3.

Ex. Let $w: \mathbb{N} \to (0, \infty)$ and set

$$\langle f, g\rangle_w = \sum_{n=1}^{\infty} f(n)\overline{g(n)}\,w(n) \quad \text{for all } f, g: \mathbb{N} \to \mathbb{C} \text{ with}$$

$$\|f\|_w, \|g\|_w < +\infty \quad (\text{here, } \|f\|_w^2 = \langle f, f\rangle_w)$$

$\ell^2_w(\mathbb{N}) = \{ f: \mathbb{N} \to \mathbb{C} : \|f\|_w < +\infty \}$.

Show that $(\ell^2_w(\mathbb{N}), \langle \cdot, \cdot \rangle_w)$ is a pre-Hilbert space.

Lemma 1 $\rightsquigarrow$ $\forall f, g \in \ell^2_w(\mathbb{N})$

$$\left| \sum_n f(n)\overline{g(n)}\,w(n) \right| \le \left(\sum_n |f(n)|^2 w(n) \right)^{1/2} \left(\sum_n |g(n)|^2 w(n) \right)^{1/2}$$

Bernstein's Lemma

Problem: When can we ensure that $f \in \mathcal{P}_1$ satisfies $\sum_m |c_f(m)| < +\infty$?

A simple instance:

Suppose $f \in \mathcal{P}_1$ is differentiable. Then $\sum_{m=-\infty}^{\infty} |c_f(m)| \leq \left|\int_0^1 f(t)\,dt\right| + \frac{2}{\sqrt{3}}\left(\int_0^1 |f'(t)|^2 dt\right)^{1/2}$.

Proof:

Recall that $c_{f'}(m) = 2\pi i m \cdot c_f(m) \quad \forall m \in \mathbb{Z}$

$$\rightsquigarrow \sum_{m=-\infty}^{\infty} |c_f(m)| = |c_f(0)| + \sum_{m \neq 0} \frac{1}{2\pi |m|} |c_{f'}(m)|$$

$$\underset{\text{Cauchy-Schwarz}}{\leq} \left|\int_0^1 f(t)\,dt\right| + \frac{1}{2\pi}\Big(\underbrace{\sum_{m\neq 0}\frac{1}{|m|^2}}_{\substack{2\cdot\frac{\pi^2}{6} \\ \text{(from Module 1)}}}\Big)^{1/2} \cdot \underbrace{\Big(\sum_{m\neq 0} |c_{f'}(m)|^2\Big)^{1/2}}_{\leq \left(\int |f'(t)|^2 dt\right)^{1/2} \text{ (Bessel's inequality)}}$$

$$\leq \left|\int_0^1 f(t)\,dt\right| + \frac{1}{2\sqrt{3}}\left(\int |f'(t)|^2 dt\right)^{1/2}. \quad ⨳$$

In many cases, the assumption that f is differentiable is too strong.

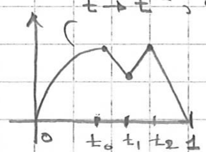
The following weakening is quite natural:

If $f \in \mathcal{P}_1$, its module of continuity is defined as

$$\omega_f(t) = \max\{|f(x) - f(y)| : |x-y| \leq t\}, \quad \text{for } t > 0.$$

Def. If $\alpha > 0$, we say that $f \in \mathcal{P}_1$ is α-Hölder if $\omega_f(t) \leq L_f \cdot t^\alpha$ for some $L_f > 0$, and for all $t > 0$.

Ex. $t \mapsto t^\alpha$, $\alpha < 1$



is α-Hölder (not differentiable at $0, t_0, t_1, t_2$)

Theorem (Bernstein's Lemma)

Suppose that $f \in \mathcal{P}_1$ satisfies $\int_0^1 |f(t+h) - f(t)|^2 dt \leq L_f^2 |h|^{2\alpha}$ $(*)$

for all $|h| < 1$ (e.g. if f is α-Hölder).

If $\alpha > 1/2$, then $\sum_m |c_f(m)| \leq \left|\int f(t)\,dt\right| + \frac{1}{\sqrt{3}}\left(\frac{2\pi}{3}\right)^\alpha \left(\frac{2^{\alpha - 1/2}}{2^{\alpha-1/2} - 1}\right) L_f$.

Cor. If $f \in \mathcal{P}_1$ is α-Hölder for some $\alpha > 1/2$, then $\sum_m |c_f(m)| < +\infty$.

Proof of Bernstein's Lemma

$$\sum_{m=-\infty}^{\infty} |c_f(m)| = |c_f(0)| + \sum_{k\geq 0}\Big(\underbrace{\sum_{2^k\leq |m|<2^{k+1}} |c_f(m)|}_{I_k}\Big)$$

Estimate I_k:

Step 1: $|e^{2\pi i m\cdot h_k}-1| \geq \sqrt{3} \quad \forall\ 2^k\leq|m|<2^{k+1}$,

where $h_k = \frac{2\pi}{3}\cdot 2^{-k}$.

Step 2: $I_k = \sum_{2^k\leq m<2^{k+1}} |c_f(m)| \leq \frac{1}{\sqrt{3}}\sum_{2^k\leq m<2^{k+1}} \left|\left(e^{2\pi i m\cdot h_k}-1\right)c_f(m)\right|$

$$\underset{\text{(Cauchy-Schwarz)}}{\leq} \frac{1}{\sqrt{3}}\left(\sum_{2^k\leq m<2^{k+1}} \left|\left(e^{2\pi i m h_k}-1\right)c_f(m)\right|^2\right)^{1/2}\cdot \underbrace{\left(2^{k+1}-2^k\right)^{1/2}}_{=2^{k/2}} \qquad (**)$$

Step 3: Set $g_h(t) = f(t+h)-f(t)$, so that

$c_{g_h}(m) = \left(e^{2\pi i h m}-1\right)c_f(m) \quad \forall m\in\mathbb{Z}$.

$\rightsquigarrow \forall k\geq 0 \quad I_k \leq \frac{2^{k/2}}{\sqrt{3}}\left(\sum_{2^k\leq|m|<2^{k+1}} |c_{g_{h_k}}(m)|^2\right)^{1/2}$

$$\leq \frac{2^{k/2}}{\sqrt{3}}\left(\sum_m |c_{g_{h_k}}(m)|^2\right)^{1/2}$$

$$\underset{\text{Bessel's inequality}}{\leq} \frac{2^{k/2}}{\sqrt{3}}\left(\int_0^1 |g_{h_k}(t)|^2\,dt\right)^{1/2}$$

By $(*)$, $\left(\int_0^1 |g_{h_k}(t)|^2\,dt\right)^{1/2} \leq L_f\,|h_k|^\alpha = \frac{2\pi}{3}L_f\cdot 2^{-k\alpha}$, and thus

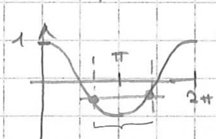
$$I_k \leq \frac{2\pi}{3\sqrt{3}}L_f\cdot 2^{-k(\alpha-1/2)} \qquad \forall k\geq 0$$

$$\rightsquigarrow \sum_{m=-\infty}^{\infty}|c_f(m)| = |c_f(0)| + \sum_{k\geq 0} I_k \leq \left|\int_0^1 f(t)\,dt\right| + \frac{2\pi}{3\sqrt{3}}L_f\underbrace{\sum_{k\geq 0} 2^{-k(\alpha-1/2)}}_{=\frac{1}{1-2^{-(\alpha-1/2)}} = \frac{2^{\alpha-1/2}}{2^{\alpha-1/2}-1}}$$

convergent if $\alpha > 1/2$

About Step 1: $|e^{iy}-1|\geq\sqrt{3} \iff |e^{iy}-1|^2 = 2(1-\cos y)\geq 3$

$\iff \cos y\leq -1/2$



$\iff \frac{2\pi}{3}\leq y\leq 2\pi\cdot\frac{2}{3}$ (modulo 2π)

If $y = 2\pi i m\cdot h_k$, $2^k\leq|m|<2^{k+1}$, then $\frac{2\pi}{3}\leq y<2\pi\cdot\frac{2}{3}$

$\Rightarrow |e^{2\pi i m h_k}-1|\geq\sqrt{3}$.