Exercises, Modulo II

1. Prove the inequality: $\left(\int_1^2 \frac{e^{-x}}{\sqrt{x}}\,dx\right)^2 \leq 2\left(1-1/\sqrt{2}\right)\int_1^2 e^{-2x}\sqrt{x}\,dx$

Is it an equality?

2. If $f \in \mathcal{P}_1$, and $M \geq 0$, set $f_M(x) = \sum_{|m|\leq M} c_f(m)\, e^{2\pi i m x}$.

Show that if f is twice diff., then $\forall \varepsilon > 0$

$$(*) \quad \int_0^1 |f(x) - f_M(x)|^2\,dx < \varepsilon \quad \forall M > \frac{1}{4\pi^2 \varepsilon}\int_0^1 |f''(t)|\,dt.$$

Find explicit M_ε s.t. $(*)$ holds $\forall M \geq M_\varepsilon$ when $f(x) = \dfrac{1}{1-\frac{1}{2}e^{2\pi i x}}$.

Hint: Consult the proof on p. 7.

3. For $m \in \mathbb{Z}$, set $F_m(y) = \int_0^1 e^{i(y\sin 2\pi x - 2\pi m x)}\,dx$, for $y \in \mathbb{R}$

Show that $\sum_{m=-\infty}^{\infty} |F_m(y)|^2 = 1 \quad \forall y \in \mathbb{R}$

4.* Suppose that $f \in \mathcal{P}_1$ is twice differentiable. Show that if $\int_0^1 f(t)\,dt = 0$, then: (Hint: Imitate proof of Wirtinger's inequality.)

$$\int_0^1 |f''(t)|^2\,dt + 256\pi^4 \int_0^1 |f(t)|^2\,dt \geq 32\pi^2 \int_0^1 |f'(t)|^2\,dt$$

and equality holds iff $f(t) = Ae^{4\pi i t} + Be^{-4\pi i t}$ for some $A, B \in \mathbb{C}$.

5. Let $f(x) = x(1-x)$, $x \in [0,1]$, and calculate $\sum_{m=1}^{\infty} |c_f(m)|^2$.

What is $\sum_{m=1}^{\infty} \frac{1}{m^4}$?

6. Show that $\left(\sum_{n=1}^{\infty} \frac{e^{-n}\sin nx}{n}\right)^2 \leq \frac{e^{-2}}{1-e^{-2}} \cdot \sum_{n=1}^{\infty} \frac{\sin^2 nx}{n^2} \quad \forall x \in \mathbb{R}$

For which x is this an equality?

7. Use Gram-Schmidt to produce an orthonormal basis for $V = \operatorname{span}\{1, x, x^2\} \subseteq C([0,2])$, w.r.t.

$$\langle f, g\rangle = \int_0^2 f(x)\overline{g(x)}\,dx$$

8. Suppose that $f \in \mathcal{P}_1$ is differentiable and $f(t) \neq 0 \ \forall t \in [0,1]$.

Show that $\sum_{m=-\infty}^{\infty} |c_{1/f}(m)| < +\infty$

Hint: Use Bernstein's Lemma.

9. Show that if $f, g \in \mathcal{P}_1$, then $\langle f, g\rangle = \sum_{m=-\infty}^{\infty} c_f(m)\overline{c_g(m)}$

10. Let $f(x) = \dfrac{1}{1-\frac{1}{2}e^{2\pi i x}}$. Use Parseval's formula to compute

$$\int_0^1 |f(x)|^2\,dx.$$

Solutions

1. $$\int_1^2 \frac{e^{-x}}{\sqrt{x}}\,dx = \int_1^2 \frac{e^{-x}}{x}\underbrace{\sqrt{x}}_{w(x)}\,dx \underset{\text{Cauchy-Schwarz}}{\leq} \left(\int_1^2 e^{-2x}\sqrt{x}\,dx\right)^{1/2}\underbrace{\left(\int_1^2 \frac{1}{x^2}\sqrt{x}\,dx\right)^{1/2}}$$

since $\frac{1}{x} \neq \lambda e^{-x}$, ineq. is strict.

$$= \int_1^2 \frac{dx}{x^{3/2}} = 2\left(1 - 1/\sqrt{2}\right)$$

2. $$f(x) - f_M(x) = \sum_{|m|>M} c_f(m)\, e^{2\pi i m x}$$

$$\rightsquigarrow \text{(Parseval)} \quad \int_0^1 |f(x) - f_M(x)|^2\,dx = \sum_{|m|>M} |c_f(m)|^2 =$$

$$= \sum_{|m|>M} \frac{1}{4\pi^2 |m|^2} \underbrace{\left|-4\pi^2 m^2 c_f(m)\right|^2}_{= c_{f''}(m)} = \left\{ |c_{f''}(m)| \leq \int |f''(t)|\,dt \quad \forall m \right\}$$

$$\leq \frac{\left(\int |f''(t)|\,dt\right)^2}{4\pi^2} \underbrace{\sum_{|m|>M} \frac{1}{m^2}}_{\leq \int_{M+1}^{\infty} \frac{dx}{x^2} = \frac{1}{M+1}} \leq \frac{1}{4\pi^2 (M+1)} \left(\int_0^1 |f''|\,dt\right)^2 < \varepsilon$$

(integral comparison)

$$\Rightarrow \quad M + 1 > \frac{\int_0^1 |f''|\,dt}{4\pi^2 \varepsilon} =: M_\varepsilon + 1$$

Case when $f(x) = \frac{1}{1 - \frac{1}{2} e^{2\pi i x}} = \sum_{m=0}^{\infty} \frac{1}{2^m} \cdot e^{2\pi i m x} \rightsquigarrow c_f(m) = \begin{cases} 1/2^m & m \geq 0 \\ 0 & m < 0 \end{cases}$

$$\rightsquigarrow \int_0^1 |f - f_M|^2\,dx = \sum_{m=M+1}^{\infty} \frac{1}{4^m} = \frac{1}{4^{M+1}} \underbrace{\left(\sum_{m=0}^{\infty} \frac{1}{4^m}\right)}_{= \frac{1}{1 - 1/4} = \frac{4}{3}} = \frac{1}{3 \cdot 4^M} < \varepsilon$$

$$\Rightarrow \quad 4^M > \frac{1}{3\varepsilon} \quad \Rightarrow \quad M > \frac{\ln(1/3\varepsilon)}{\ln 4}$$

3. $f_y(x) = e^{iy \sin 2\pi x}$; note $c_{f_y}(m) = F_m(y)$

Parseval: $$\underbrace{\int_0^1 |f_y(x)|^2\,dx}_{= 1 \ \forall y} = \sum_{m=-\infty}^{\infty} |F_m(y)|^2$$

4*. Parseval: $$\int_0^1 |f''(t)|^2\,dt = 16\pi^4 \sum_m m^4 \cdot |c_f(m)|^2 \qquad \left(c_{f''}(m) = -4\pi^2 m^2 c_f(m)\right)$$

$$\int_0^1 |f'(t)|^2\,dt = 4\pi^2 \sum_m m^2 \cdot |c_f(m)|^2 \qquad \left(c_{f'}(m) = 2\pi i m\, c_f(m)\right)$$

$$\left(\int_0^1 f(t)\,dt = 0\right): \quad \int_0^1 |f(t)|^2\,dt = \sum_{m \neq 0} |c_f(m)|^2$$

$$\rightsquigarrow \int_0^1 |f''(t)|^2\,dt - 4\pi^2 A \int_0^1 |f'(t)|^2 + 16\pi^4 \cdot B \int_0^1 |f(t)|^2\,dt$$

$$= 16\pi^4 \left(\sum_{m \neq 0} \underbrace{\left(m^4 - A m^2 + B\right)}_{\left(m^2 - A/2\right)^2 + B - A^2/4} |c_f(m)|^2 \right) = \{A = 8,\ B = 16\}$$

$$= 16\pi^4 \cdot \sum_{m \neq 0} \left(m^2 - 4\right)^2 |c_f(m)|^2 \geq 0 \qquad \left(= 0 \Leftrightarrow c_f(m) = 0 \ \ \forall |m| \neq 2\right)$$

$$\Leftrightarrow f(x) = \alpha e^{4\pi i x} + \beta e^{-4\pi i x}$$

5. By Exercise 8a) in Module I, $c_f(m) = \begin{cases} 1/6 & m=0 \\ -\frac{1}{2\pi^2 m^2} & m \neq 0 \end{cases}$

Parseval: $\sum_{m=-\infty}^{\infty} |c_f(m)|^2 = \frac{1}{36} + \frac{1}{4\pi^4} \sum_{m\neq 0} \frac{1}{m^4}$

$= \frac{1}{36} + \frac{1}{2\pi^4} \sum_{m=1}^{\infty} \frac{1}{m^4} = \int_0^1 |f(t)|^2 dt = \int_0^1 t^2(1-t)^2 dt$

$= \int_0^1 (t^2 - 2t^3 + t^4) dt = \frac{1}{3} - 2 \cdot \frac{1}{4} + \frac{1}{5} = \frac{10-15+6}{2\cdot 3\cdot 5} = \frac{1}{30}$

$\rightsquigarrow \sum_{m=1}^{\infty} \frac{1}{m^4} = \left(\frac{1}{30} - \frac{1}{36}\right) \cdot 2\pi^4 = \frac{6 \cdot 2\pi^4}{30 \cdot 6^2} = \frac{\pi^4}{90}$

6. $\left|\sum_{n=1}^{\infty} \frac{e^{-n} \sin nx}{n}\right| \underset{\text{Cauchy-Schwarz}}{\leq} \left(\sum_{n=1}^{\infty} e^{-2n}\right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{\sin^2 nx}{n^2}\right)^{1/2}$

$\sum_{n=1}^{\infty} e^{-2n} = \sum_{n=0}^{\infty} e^{-2n} - 1 = \frac{1}{1-e^{-2}} - 1 = \frac{e^{-2}}{1-e^{-2}}$

If equality, then $\lambda e^{-n} = \frac{\sin nx}{n}$ $\forall n \geq 1$
(for some x)

(for some $\lambda \in \mathbb{C}$) $\Rightarrow \begin{cases} \sin nx = 0 & \forall n \geq 1 \\ \lambda = 0 \end{cases}$

$\rightsquigarrow x = m \cdot \pi$ for some $m \in \mathbb{Z}$.

7. Set $f_0(x) = 1$, $f_1(x) = x$, $f_2(x) = x^2$.

$\varphi_0 = \frac{f_0}{\|f_0\|}$,

$\tilde\varphi_1 = f_1 - \langle f_1, \varphi_0\rangle \varphi_0 \rightsquigarrow \varphi_1 = \frac{\tilde\varphi_1}{\|\tilde\varphi_1\|}$, $\|\tilde\varphi_1\|^2 = \|f_1\|^2 - |\langle f_1, \varphi_0\rangle|^2$

$\tilde\varphi_2 = f_2 - \langle f_2, \varphi_1\rangle \varphi_1 - \langle f_2, \varphi_0\rangle \varphi_0 \rightsquigarrow \varphi_2 = \frac{\tilde\varphi_2}{\|\varphi_2\|}$, $\|\tilde\varphi_2\|^2 = \|f_2\|^2 - |\langle f_2, \varphi_1\rangle|^2 - |\langle f_2, \varphi_0\rangle|^2$

$\|f_0\| = \sqrt{2}$, $\|f_1\| = \sqrt{\frac{8}{3}}$, $\|f_2\| = \sqrt{\frac{32}{5}}$.

8. If $f \in \mathcal{P}_1$ diff and non-zero, so is $g = 1/f$. Since g is 1-Hölder,

by Bernstein: $\sum_{m=-\infty}^{\infty} |c_g(m)| < +\infty$.

10. $c_f(m) = \begin{cases} 0 & m<0 \\ 1/2^m & m \geq 0 \end{cases} \rightsquigarrow \int_0^1 |f(x)|^2 dx \underset{\text{Parseval}}{=} \sum_{m=0}^{\infty} \frac{1}{4^m} = \frac{1}{1-1/4} = \frac{4}{3}$