MODULE I

- Benford's Law and Weyl's Criterion
- Fejer's Thm and Dirac sequences.

Aim: To give an explicit example of how Fourier analysis can break down a cplx (hard) problem (establishing Benford's Law) into simpler components (Weyl's Criterion).

Recommended exercises: 1, 3, 4, 5, 6, 7, 8ab, 9, 11, 12, 13, 14, 15

["Ringed" exercises are strongly recommended ...]

Important to focus on:

- Fejer's Thm / use of Dirac sequences
- Fejer sums / Fourier series (additional material)
- Periodization (additional material).

Benford's Law

See Wikipedia for applications of Benford's Law in accounting, election fraud etc.

— * —

If $n \in \mathbb{N}$, set $(n)_1$ = first digit in base 10-expansion of n.

Ex. $(7)_1 = 7$, $(34)_1 = 3$, $(127)_1 = 1$, $(4759)_1 = 4$.

Def. A sequence (n_k) in $\mathbb{N}$ is called Benford if

$$\lim_{N \to \infty} \frac{\#\{k = 1, \dots, N : (n_k)_1 = d\}}{N} = \log_{10}\left(1 + \frac{1}{d}\right) \quad \forall d = 1, 2, \dots, 9.$$

Many sequences which arise "in nature" are Benford (see Wikipedia).

Non-example: $n_k = k \ \forall k$ (why?)

Question: How to prove that a given sequence is Benford?

Thm (Weyl's Criterion, weak version)

A sequence $(n_k) \subset \mathbb{N}$ is Benford if

$$(*) \quad \lim_N \frac{1}{N} \sum_{k=1}^{N} n_k^{2\pi i m / \ln 10} = 0 \quad \forall m \neq 0$$

Ex. $n_k = 2^{k-1}$, $k \geq 1$.

$$\forall m \neq 0 \quad (*) = \lim_N \frac{1}{N} \sum_{k=0}^{N-1} 2^{2\pi i k m / \ln 10} = \{\text{geom. sum}\}$$

$|\cdot| \leq 2 \ \forall N$

$$= \lim_N \frac{1}{N} \left(\frac{2^{\frac{2\pi i N m}{\ln 10}} - 1}{2^{\frac{2\pi i m}{\ln 10}} - 1} \right) = 0$$

$\neq 0$ ($\Leftrightarrow$ $m \log_{10} 2 \in \mathbb{Z}$ impossible!) since $\log_{10} 2$ irrational

Cor. (Gelfand's problem)

Is there an exponent of 2 with 9 as leading digit in base 10-exp?

$(2^0)_1 = 1$, $(2^1)_1 = 2$, $(4)_1 = 4$, $(16)_1 = 1$, $(32)_1 = 3$, $(64)_1 = 6$

$(128)_1 = 1$, $(256)_1 = 2$, ...

By previous example: (2^{k-1}) is Benford, hence

$$\lim_N \frac{\#\{k = 0, \dots, N-1 : (2^k)_1 = 9\}}{N} = \log_{10}\left(1 + \frac{1}{9}\right) \approx 0{,}046\ldots > 0$$

$\rightsquigarrow$ There are infinitely many exponents of 2 with leading digit 9.

(Smallest: 2^{54})

Weyl's Criterion, week version

$$n = \sum_{\ell=0}^{q_n} d_\ell \cdot 10^\ell \quad , \quad d_\ell \in \{0,1,\dots,9\}, \quad d_{q_n} \neq 0.$$

Obs 1: $(n)_1 = d \iff d \cdot 10^{q_n} \leq n < (d+1)\cdot 10^{q_n}$

$\iff \log_{10} d + q_n \leq \log n < \log_{10} d + (q_n + 1)$

$\iff \log n \bmod 1 \in [\log_{10} d, \log_{10}(d+1)) \subset [0,1)$.

Set $I_d = [\log_{10} d, \log_{10}(d+1))$ – interval of length $\log_{10}(d+1) - \log_{10} d$
$= \log_{10}\left(1 + \frac{1}{d}\right)$.

$x_k = \log_{10} n_k \pmod 1 \in [0,1)$.

Obs 2: $\frac{\#\{k = 1,\dots,N : (n_k)_1 = d\}}{N} = \frac{1}{N}\sum_{k=1}^{N} \chi_{I_d}(x_k)$

Def: A sequence $(x_k) \subset [0,1)$ is equidistributed if

(**) $\lim_N \frac{1}{N}\sum_{k=1}^{N} f(x_k) = \int_0^1 f(x)\,dx$ $\forall$ Riemann-integrable $f : [0,1] \to \mathbb{C}$

Rmk: Since $f = \chi_{I_d}$ Riemann-integrable $\forall d = 1,\dots,9$, we see that: ($\log_{10} n_k \bmod 1$) equidist. $\Rightarrow$ (n_k) Benford.

Thm (Weyl's Criterion)

A sequence $(x_k) \subset [0,1)$ is equidistributed if and only if

(***) $\lim_N \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i m x_k} = 0 \quad \forall m \neq 0$

Obs: $x_k = \log_{10} n_k$ equidistributed iff

$$\lim_N \frac{1}{N}\sum_{k=1}^{N} \underbrace{e^{2\pi i m \log_{10} n_k}}_{= n_k^{\frac{2\pi i m}{\log 10}}} = 0 \quad \forall m \neq 0$$ (→ Weak version of Weyl's criterion)

[Only "if"-direction is non-trivial: Clearly $f(x) = e^{2\pi i m x}$ is Riemann integrable $\forall m$, and $\int_0^1 e^{2\pi i m x}\,dx = 0 \quad \forall m \neq 0$.]

Proof structure (Weyl's criterion):

1) (**) $\iff$ (**) holds $\forall$ cts $f : [0,1] \to \mathbb{C}$ with $f(0) = f(1)$

2) $\iff$ —"— $\forall$ trig. polynomials $f : [0,1] \to \mathbb{C}$

$\iff$ (***)

Most important equivalence

Weyl's Criterion

Let $(x_k) \subset [0,1)$ and assume:

$$\lim_N \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i m x_k} = 0 \quad \forall m \neq 0. \qquad (*)$$

Want to show: $\forall f : [0,1] \to \mathbb{C}$ Riemann-integrable

$$\lim_N \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_0^1 f(x)\,dx \qquad (*)_f$$

Def. • $f(x) = \sum_{m=-M}^{M} c(m)\, e^{2\pi i m x}$, for some $M \geq 0$ and $c : [-M,M] \to \mathbb{C}$, is called a trigonometric polynomial (of order M).

• $T_1 = \{ f \in C([0,1]) : f \text{ trig. poly} \}$

$P_1 = \{ f \in C([0,1]) : f(0) = f(1) \}$

($C([0,1])$ = space of cnts fcns $[0,1] \to \mathbb{C}$)

Lemma 1: $(*) \Rightarrow (*)_f \quad \forall f \in T_1$

Proof. Easy (exercise).

Lemma 2: $(*)_f \ \forall f \in T_1 \Rightarrow (*)_f \ \forall f \in P_1$

Proof:

Thm (Fejer's thm, weak version) $T_1 \subset P_1$ is dense,

i.e. $\forall f \in P_1 \ \forall \varepsilon > 0 \ \exists p \in T_1$ s.t.

$$\|f - p\|_\infty := \max_{x \in [0,1]} |f(x) - p(x)| < \varepsilon.$$

Assume Thm: Fix $f \in P_1$ and $\varepsilon > 0$. Pick $p \in T_1$ s.t. $\|f-p\|_\infty < \varepsilon/3$.

$$(**)_N \quad \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \int_0^1 f(x)\,dx = \frac{1}{N} \sum_{k=1}^{N} p(x_k) - \int_0^1 p(x)\,dx \; +$$

$$+ \int_0^1 (p(x) - f(x))\,dx + \frac{1}{N} \sum_{k=1}^{N} (f(x_k) - p(x_k)), \quad \forall N$$

By Lemma 1, $\exists N_0 \geq 1$ s.t. $\left| \frac{1}{N} \sum_{k=1}^{N} p(x_k) - \int_0^1 p(x)\,dx \right| < \varepsilon/3, \ \forall N \geq N_0$

$\rightsquigarrow \ |(**)_N| < \varepsilon/3 + 2\|p - f\|_\infty < \varepsilon, \quad \forall N \geq N_0$

Since $\varepsilon > 0$ is arbitrary: $\quad \lim_N \frac{1}{N} \sum_{k=1}^{N} f(x_k) = \int_0^1 f(x)\,dx$

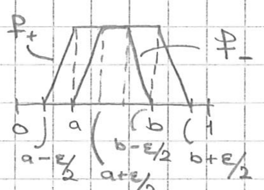
Weyl's Criterion (cont'd)

Lemma 3 $(*)_f \ \forall f \in \mathcal{P}_1 \Rightarrow (*)_f \ \forall f = \chi_I$, $I \subseteq [0,1)$ interval.

Proof. Fix $I = [a,b)$ and $0 < \varepsilon < \min(a, 1-b, (b-a)/2)$

Then: $\exists f_-, f_+ \in \mathcal{P}_1$ s.t. $f_- \le f \le f_+$ and $\int_0^1 (f_+ - f_-)\,dx < \varepsilon$.

Indeed:



(Case $b=1$ is a bit trickier, but can be solved exactly as below)

$$\rightsquigarrow \quad \frac{1}{N}\sum_{k=1}^{N} f_-(x_k) \le \frac{1}{N}\sum_{k=1}^{N} f(x_k) \le \frac{1}{N}\sum_{k=1}^{N} f_+(x_k)$$

$$\rightsquigarrow \quad \underline{\lim_N}\ \frac{1}{N}\sum_{k=1}^{N} f(x_k) \ge \int_0^1 f_-(x)\,dx = \int_0^1 f(x)\,dx - \int_0^1 (f - f_-)\,dx$$

$$\ge \int_0^1 f(x) - \varepsilon.$$

$$\overline{\lim_N}\ \frac{1}{N}\sum_{k=1}^{N} f(x_k) \le \int_0^1 f_+(x)\,dx = \int_0^1 f(x) + \int_0^1 (f_+ - f)\,dx$$

$$\le \int_0^1 f(x) + \varepsilon$$

Since $\varepsilon > 0$ arbitrary: $\quad \underline{\lim_N}\ \frac{1}{N}\sum_{k=1}^{N} f(x_k) = \overline{\lim_N}\ \frac{1}{N}\sum_{k=1}^{N} f(x_k) = \int_0^1 f(x)\,dx$

$\between$

Lemma 4 $(*)_f \ \forall f = \chi_I$, $I \subseteq [0,1)$ interval $\Rightarrow (*)_f \ \forall f$ Riemann-int.

Proof: By def: f is Riemann-int if $\forall \varepsilon > 0 \ \exists f_-, f_+$ step fcns (= finite sums of indicator fcns of intervals) s.t. $f_- \le f \le f_+$ and $\int_0^1 (f^+ - f^-)\,dx < \varepsilon$. Proof is now exactly as in Lemma 3.

$\between$

$\rightsquigarrow$ Weyl's Criterion

Remains: Show Fejér's thm (weak version).

Fejer's Thm, weak version

Recall: $\mathcal{P}_1 = \{ f \in C([0,1]) : f(0) = f(1) \} \simeq \{ f \in C(\mathbb{R}) : f \text{ 1-periodic} \}$

Let $f \in \mathcal{P}_1$, and define:

- $c_f(m) = \int_0^1 f(x) e^{-2\pi i m x}\, dx$, $m \in \mathbb{Z}$ [Fourier coefficients of f]
- $s_M(f)(x) = \sum_{m=-M}^{M} c_f(m) e^{2\pi i m x}$, $M \geq 0$ [Dirichlet sums of f]
- $\sigma_N(f)(x) = \frac{1}{N} \sum_{M=0}^{N-1} s_M(f)$, $N \geq 1$ [Fejér sums of f]

Obs: $s_M(f)$ trig. polynomial of order M

$\sigma_N(f)$ trig. polynomial of order $N-1$,

so $s_M(f), \sigma_N(f) \in T_1$ $\forall M, N$.

$$\boxed{\begin{array}{l} \text{Fejer's Thm} \quad \forall f \in \mathcal{P}_1, \quad \lim_N \| \sigma_N(f) - f \|_\infty = 0 \\ \left(\text{In particular, } \sigma_N(f)(x) \to f(x) \quad \forall x \in [0,1] \right) \end{array}}$$

Rmk: Since $\sigma_N(f) \in T_1$, this implies the weak version on p. 3.

Rmk: $\exists f \in \mathcal{P}_1$ s.t. $\| s_M(f) - f \|_\infty \not\to 0$ as $M \to \infty$ so the analogue of Fejer's Thm for Dirichlet sums is not true (More about this later...)

Towards the proof of Fejer's Thm:

- $$\sigma_N(f)(x) = \frac{1}{N} \sum_{M=0}^{N-1} \left(\sum_{m=-M}^{M} c_f(m) e^{2\pi i m x} \right)$$
$$= \frac{1}{N} \sum_{n=0}^{N-1} (N - |n|) c_f(n) e^{2\pi i n x} = \sum_{n=-(N-1)}^{N-1} \left(1 - \frac{|n|}{N}\right) c_f(n) e^{2\pi i n x}$$

- Set $F_N(x) = \sum_{n=-(N-1)}^{N-1} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}$, and note that

"convolution"

$$(F_N * f)(x) := \int_0^1 F_N(y) f(x-y)\, dy = \sum_{n=-(N-1)}^{(N-1)} \left(1 - \frac{|n|}{N}\right) \int_0^1 \underbrace{e^{2\pi i n y} f(x-y)}_{= e^{-2\pi i n (x-y)} \cdot e^{2\pi i n x} \cdot f(x-y)}\, dy$$

f 1-periodic

$$= \sum_{n=-(N-1)}^{N-1} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} \underbrace{\int_0^1 e^{-2\pi i n (x-y)} f(x-y)\, dy}_{= c_f(n)}$$

$$= \sigma_N(f)(x), \quad \forall N.$$

Fejer's Thm

If $f, g \in \mathcal{R}_1$, their <u>convolution</u> is defined by

$$(f*g)(x) = \int_0^1 f(y)\, g(x-y)\, dy,$$

(where we think of f, g as 1-periodic ent. fcns on $\mathbb{R}$)

<u>Fejer kernel</u>: $\quad F_N(x) = \sum_{n=-(N-1)}^{N-1} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} = \frac{1}{N}\left(\frac{\sin \pi N x}{\sin \pi x}\right)^2 \quad (*)$

(Exercise 4)

<u>Recall</u>: Fejer sum $\sigma_N(f) = F_N * f$.

<u>Def.</u> A sequence (K_N) of Rieman-integrable, 1-periodic fcns on $\mathbb{R}$ is called <u>Dirac</u> if

i) $\int_0^1 K_N(x)\, dx = 1$

ii) $C_0 := \sup_N \int |K_N(x)|\, dx < +\infty$

iii) $\forall\, 0 < \delta < 1/2 \qquad \int_\delta^{1-\delta} |K_N(x)|\, dx \to 0$, as $N \to \infty$.

<u>Lemma 1</u> Fejer's kernel (F_N) is Dirac.

<u>Lemma 2</u> If (K_N) is Dirac and $f \in \mathcal{R}_1$, then $\|K_N * f - f\|_\infty \to 0$.

• <u>Obs</u>: Lemma 1 + Lemma 2 $\Rightarrow$ Fejer's Thm (p. 5)

• <u>Obs</u>: If $K_N \geq 0$, then ii) follows from i).

<u>Proof of Lemma 1</u>:

i) $\int_0^1 F_N(x)\, dx = \sum_{n=-(N-1)}^{N-1} \left(1 - \frac{|n|}{N}\right) \underbrace{\int_0^1 e^{2\pi i n x}\, dx}_{= \begin{cases} 0 & \forall n \neq 0 \\ 1 & n = 0 \end{cases}} = 1 \quad \forall N$

ii) By $(*)$, $F_N \geq 0$, so ii) automatic

iii) If $0 < \delta \leq x \leq 1-\delta < 1$, then $|F_N(x)| \leq \frac{1}{N} \frac{1}{\sin^2 \pi \delta} \quad \forall N$

and thus:

$\rightsquigarrow \int_\delta^{1-\delta} |F_N(x)|\, dx \leq \frac{1}{N}\left(\frac{1-2\delta}{\sin^2 \pi \delta}\right) \to 0$, as $N \to \infty$.

$\boxtimes$

Fejer's Thm (cont'd)

Proof of Lemma 2:

Pick $f \in \mathcal{P}_1 \subset C([0,1])$. Since $[0,1]$ is compact, f is uniformly continuous, i.e. $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

If (K_N) is Dirac, then:

- By (i): $(K_N * f)(x) - f(x) = \int_0^1 K_N(y)\,(f(x-y) - f(x))\,dy$
- Fix $\varepsilon > 0$, and pick $\delta > 0$ s.t. $|y| < \delta \Rightarrow |f(x-y) - f(x)| < \varepsilon$ (unif. cont. of f)

$$\rightsquigarrow (K_N * f)(x) - f(x) = \int_0^\delta + \int_{1-\delta}^1 + \int_\delta^{1-\delta} K_N(y)\,(f(x-y) - f(x))\,dy$$

$$\rightsquigarrow | \qquad | \leq 2\varepsilon \cdot \int_0^1 |K_N(y)|\,dy + 2\|f\|_\infty \int_\delta^{1-\delta} |K_N(y)|\,dy$$

$$\rightsquigarrow \overline{\lim_N} \, |(K_N * f)(x) - f(x)| \leq 2C\varepsilon \qquad \forall x$$

Since $\varepsilon > 0$ is arbitrary, $K_N * f \to f$ uniformly $\quad$ ⌧.

- **Summary:**

Fejer's Thm $\rightsquigarrow$ Trig. polynomials are dense in $\mathcal{P}_1$

$\rightsquigarrow$ Weyl's Criterion

$\rightsquigarrow$ Weyl's Criterion, weak version ($\sim$ Benford's Law)

- **Important concepts to focus on:**
  - Fourier coefficients and Fejer sums.
  - Dirac sequences
  - Equidistribution.

Additional material, Module I

Recall: If $f \in \mathcal{P}_1$, its Fourier coefficients $(c_f(m))$ are defined as

$$c_f(m) = \int_0^1 f(x) e^{-2\pi i m x}\, dx, \quad m \in \mathbb{Z}. \qquad (*)$$

(In particular, $|c_f(m)| \leq \int_0^1 |f(x)|\,dx \ \forall m \in \mathbb{Z}$.)

Rmks:

1) Since f and $e^{-2\pi i m x}$ are 1-periodic ($m \in \mathbb{Z}$), they are uniquely determined on any interval of length 1, e.g. $[-1/2, 1/2)$, and thus

$$c_f(m) = \int_{-1/2}^{1/2} f(x) e^{-2\pi i m x}\, dx = \int_{-1/4}^{3/4} f(x) e^{-2\pi i m x} = \dots \quad \forall m \in \mathbb{Z}. \qquad (**)$$

2) $\forall m$, the map $\mathcal{P}_1 \ni f \mapsto c_f(m)$ is linear, i.e.

- $c_{\lambda \cdot f}(m) = \lambda \cdot c_f(m) \quad \forall m \ \forall f \in \mathcal{P}_1 \ \forall \lambda \in \mathbb{C}$
- $c_{f+g}(m) = c_f(m) + c_g(m) \quad \forall m \ \forall f, g \in \mathcal{P}_1$

3) $\boxed{\forall f, g \in \mathcal{P}_1 \quad c_{f*g}(m) = c_f(m)\, c_g(m) \quad \forall m \in \mathbb{Z}}$ "Fundamental property of Fourier coefficients"

Proof:

$$c_{f*g}(m) = \int_0^1 (f*g)(x) e^{-2\pi i m x}\, dx = \int_0^1\int_0^1 f(y) g(x-y) e^{-2\pi i m (y + x - y)}\, dy\, dx$$

$$= \{\text{change order of integration}\} = \int_0^1 f(y) e^{-2\pi i y}\Big(\underbrace{\int_0^1 g(x-y) e^{-2\pi i (x-y) m}\, dx}_{= \int_{-y}^{1-y} g(x) e^{-2\pi i m x}\, dx = c_g(m) \ \forall y \text{ by } (**)}\Big)\, dy$$

$$= c_f(m)\, c_g(m) \quad \forall m.$$

4) Set $\mathcal{P}_1^{(k)} = \{ f \in C^k(\mathbb{R}) : \text{1-periodic}\}$, $k \geq 1$.

$\boxed{\text{If } f \in \mathcal{P}_1^{(1)}, \text{ then } c_{f'}(m) = 2\pi i m\, c_f(m) \ \forall m \in \mathbb{Z}}$

Proof: $c_{f'}(m) = \int_0^1 f'(x) e^{-2\pi i m x}\, dx = \{\text{partial integration}\}$

$$= \left[f(x) e^{-2\pi i m x}\right]_0^1 - (-2\pi i m) \int_0^1 f(x) e^{-2\pi i m x}\, dx = 2\pi i m \cdot c_f(m), \quad \forall m$$

5) [Weierstraß M-test] Sps $\varphi_n : [0,1] \to \mathbb{C}$ s.t. $\sum_n \|\varphi_n\|_\infty < +\infty$.

Then $f(x) = \sum_n \varphi_n(x)$ exists $\forall x \in [0,1]$ and

$$\sum_{n=1}^N \varphi_n(x) \longrightarrow f(x) \ \text{uniformly on } [0,1].$$

In particular, $f \in C([0,1])$.