"Fourier series of f": $f(x) = \sum_{m=-\infty}^{\infty} c_f(m)\, e^{2\pi i m x} \quad \forall x$

6) If $\sum_{m=-\infty}^{\infty} |a_m| < +\infty$, then $f(x) = \sum_{m=-\infty}^{\infty} a_m e^{2\pi i m x}$ belongs to $\mathcal{P}_1$

and $c_f(m) = a_m \quad \forall m \in \mathbb{Z}$. (see also Exercise 13)

<u>Proof</u>: By Weierstraß M-test (5) above), f is continuous.

Since each added is 1-periodic, so is f; hence: $f \in \mathcal{P}_1$

Also, if we set $f_N(x) = \sum_{m=-N}^{N} a_m e^{2\pi i m x}$, then $f_N \to f$ uniformly on $[0,1]$.

<u>Hence</u>: $\forall m \in \mathbb{Z}$ $\quad \lim_N |c_f(m) - c_{f_N}(m)| \leq \lim_N \underbrace{\int_0^1 |f(x) - f_N(x)|\,dx}_{\leq \|f - f_N\|_\infty \to 0} = 0$ $\qquad (***)$

<u>Note</u>: $c_{f_N}(n) = \sum_{m=-N}^{N} a_m \underbrace{\int_0^1 e^{2\pi i (m-n)x}\,dx}_{= \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases}}$

$$= \begin{cases} 0 & \text{if } |n| > N \\ a_n & \text{if } |n| \leq N \end{cases} \longrightarrow a_n, \text{ as } N \to \infty.$$

$\rightsquigarrow \quad c_f(m) = a_m \quad \forall m \in \mathbb{Z}$ $\qquad \S$

7) [Periodization] Sps. $F: \mathbb{R} \to \mathbb{C}$ is continuous, and $\exists\, C_1 < +\infty$

s.t. $|F(x)| \leq \dfrac{C_1}{1+|x|^2} \quad \forall x \in \mathbb{R}$. $\qquad (****_1)$

<u>Then</u>:

i) $f(x) := \sum_{n=-\infty}^{\infty} F(x+n)$ exists $\forall x \in [0,1]$ and belongs to $\mathcal{P}_1$

ii) $c_f(m) = \hat{F}(m) \quad \forall m \in \mathbb{Z}$, where

$\hat{F}(\xi) = \int_{-\infty}^{\infty} F(x) e^{-2\pi i x \xi}\,dx$, $\quad \xi \in \mathbb{R}$ (Fourier transform of F)

iii) If $\exists\, C_2$ s.t. $|\hat{F}(\xi)| \leq \dfrac{C_2}{1+|\xi|^2} \quad \forall \xi \in \mathbb{R}$ $\qquad (****_2)$

then $\boxed{\sum_m \hat{F}(m) e^{2\pi i m x} = \sum_n F(x+n) \quad \forall x \in [0,1]}$ "Poisson's Summation Formula"

<u>Proof</u>: i) Set $\varphi_n(x) = F(x+n)$, $\quad x \in [0,1]$

$\|\varphi_n\|_\infty = \max_{x \in [n,n+1]} |F(x)| \leq \dfrac{C_1}{1+|n|^2} \quad \forall n \rightsquigarrow \sum_n \|\varphi_n\|_\infty < +\infty$

$\rightsquigarrow$ [Weierstraß M-test] $\quad f(x) = \sum_{n=-\infty}^{\infty} \varphi_n(x)$ exists and is continuous on $[0,1]$

Since the sum is over all of $\mathbb{Z}$, f is 1-periodic $\rightsquigarrow f \in \mathcal{P}_1$

Also, $f_N(x) = \sum_{n=-N}^{N} F(x+n)$ converges <u>uniformly</u> to f.

ii) $c_{f_N}(m) = \sum_{n=-N}^{N} \underbrace{\int_0^1 F(x+n) e^{-2\pi i m x}\,dx}_{\int_n^{n+1} F(x) e^{-2\pi i m (x-n)}\,dx} = \sum_{n=-N}^{N} \int_n^{n+1} F(x) e^{-2\pi i m x}\,dx =$ (since $e^{2\pi i m \cdot n} = 1 \quad \forall m,n \in \mathbb{Z}$)

$= \int_{-N}^{N+1} F(x) e^{-2\pi i m x}$

Exactly as in 6) *** above: $c_{f_N}(m) \to c_f(m) \ \forall m \in \mathbb{Z}$.

Since F is Riemann integrable on $(-\infty, \infty)$,

$$c_{f_N}(m) = \int_{-N}^{N+1} F(x)\, e^{-2\pi i m x}\, dx \longrightarrow \int_{-\infty}^{\infty} F(x)\, e^{-2\pi i m x}\, dx = \hat{F}(m)$$

(ii) If $|\hat{F}(\xi)| \leq \dfrac{C_2}{1+|\xi|^2} \ \forall \xi \in \mathbb{R}$, then $\sum_m |c_f(m)| < +\infty$

and thus:

$$f(x) = \sum_{m=-\infty}^{\infty} c_f(m)\, e^{2\pi i m x} = \sum_{m=-\infty}^{\infty} \hat{F}(m)\, e^{2\pi i m x} \quad \forall x \in [0,1].$$

(by 6 above)

8) [Application of Poisson's summation formula]

Let $t > 0$ and set $F(x) = e^{-t|x|}$ (condition (****$_1$) is obvious).

- $$\hat{F}(\xi) = \int_{-\infty}^{\infty} e^{-t|x|} \cdot e^{-2\pi i \xi x}\, dx = \int_{-\infty}^{0} e^{(t-2\pi i \xi)x}\, dx + \int_{0}^{\infty} e^{(-t-2\pi i \xi)x}\, dx$$

$$= \frac{1}{t-2\pi i \xi} - \frac{1}{-t-2\pi i \xi} = \frac{1}{t+2\pi i \xi} + \frac{1}{t-2\pi i \xi} = \frac{2t}{t^2+4\pi^2\xi^2}$$

- $$f(0) = \sum_{n=-\infty}^{\infty} e^{-t|n|} = 2\sum_{n=0}^{\infty} e^{-tn} - 1 = \{\text{geom. sum}\}$$

$$= \frac{2}{1-e^{-t}} - 1 = \frac{1+e^{-t}}{1-e^{-t}}$$

Poisson summation at $x=0$:

$$\boxed{\frac{1+e^{-t}}{1-e^{-t}} = \sum_{m=-\infty}^{\infty} \frac{2t}{t^2+4\pi^2 m^2} = \frac{2}{t} + 2\sum_{m=1}^{\infty} \frac{2t}{t^2+4\pi^2 m^2}} \quad \forall t > 0$$

(What happens when $t \to 0^+$?)

(Later we shall consider $F(x) = e^{-tx^2} \rightsquigarrow$ Jacobi's Θ-fcn).

9) [Convergence of Fejer sums for piecewise continuous 1-periodic fcns]

If $f: \mathbb{R} \to \mathbb{C}$ 1-periodic (Riemann-integrable), $x_0 \in \mathbb{R}$ s.t. $\lim_{x \to x_0^\pm} f(x) = f(x_0^\pm)$ exist (for $\pm$)

then $\sigma_N(f)(x_0) \to \frac{1}{2}\left(f(x_0^+) + f(x_0^-)\right)$, as $N \to \infty$.

(Proof is very close to "Fejer's thm" – is omitted here).

(Also compare with Folland, Thm 2.1 (for Dirichlet sums))

Exercises (Module 1) (* = Harder problem)

1. Is the sequence $x_k = \log k \pmod 1$ equidistributed?

2. Benford's Law can be formulated for any base (not necessarily 10)
   - What is the corresponding Weyl criterion?
   - With respect to which bases is the sequence $n_k = 2^k$ Benford?

3. Show that if (x_k) is equidistributed, then $y_k = c + x_k \pmod 1$ is equidistributed $\forall c \in \mathbb{R}$.

4. Prove the identity: $F_N(x) = \frac{1}{N}\left(\frac{\sin \pi N x}{\sin \pi x}\right)^2 \quad \forall x \in [0,1) \ \forall N \geq 2$

[Hint: What is $\sin^2 \pi x \cdot F_N(x)$?]

5. Show that if $f \in \mathcal{P}_1$ and $c_f(m) = 0 \ \forall m \in \mathbb{Z}$, then $f \equiv 0$.

6. [Riemann-Lebesgue's Lemma, weak version] Show that if $f \in \mathcal{P}_1$, then $\lim_{|m| \to \infty} c_f(m) = 0$.

[Hint: Trivial for $f \in T_1$, use Fejer's thm to conclude $\forall f \in \mathcal{P}_1$]

7. [Fejer's Lemma, weak version] Show that if $f, g \in \mathcal{P}_1$, then $\lim_{n \to \infty} \int_0^1 f(x)\, g(n \cdot x)\, dx = \int_0^1 f(x)\, dx \cdot \int_0^1 g(x)\, dx$

[Hint:]

8. a) Compute the Fourier coefficients for $f(x) = x(1-x)$, $x \in [0,1]$ (clearly, $f \in \mathcal{P}_1$)

   b) Show that
   - $\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}$ [Mengoli's Basel problem] ((solved by Euler 1734) (see also Wikipedia))
   - $\sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} = -\frac{\pi^2}{12}$

[Hint: Use Fejer's thm at $x = 0$ and $x = 1/2$] (alt. B) in "Additional methods")

9. Calculate $(c_f(m))$ for: $f(x) = \frac{1}{1 - \frac{1}{2}\cos 2\pi x}$, $x \in [0,1]$. What is $c_f(1)$?

[Hint: Use B) in "Additional material", and binomial formula]

10*. Let $h(x) = x(1-x)$ and $g(x) = \frac{1}{1 + \frac{1}{3} e^{2\pi i x}}$ $x \in [0,1]$.

Find $f \in \mathcal{P}_1$ s.t. $g = h * f$.

[Hint: Use 3) in "Additional material" to find $c_f(m)$ - Also, $(c_h(m))$ is computed in 8a) above.]

11. Express $f(x) = \begin{cases} x\left(\frac{1}{2} - x\right) & 0 \leq x < \frac{1}{2} \\ -x\left(\frac{1}{2} - x\right) & -\frac{1}{2} < x \leq 0 \end{cases}$ (odd fcn.)

on the form $\sum_{n=1}^{\infty} a_n \sin 2\pi n x$ (for some sequence (a_n)).

12. Show that if $f \in \mathcal{S}^{(2)}$, then $f(x) = \underbrace{\sum_{m=-\infty}^{\infty} c_f(m)\, e^{2\pi i m x}}_{\text{absolutely convergent sum.}}$ $\forall x \in [0,1]$

13. Let $f \in \mathcal{S}_1$ and suppose that $\sum_{m=-\infty}^{\infty} |c_f(m)| < +\infty$.

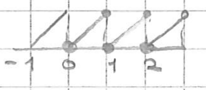
Show that $f(x) = \underbrace{\sum_{m=-\infty}^{\infty} c_f(m)\, e^{2\pi i m x}}_{= g(x)}$.

[Hint: Show that $g(x)$ is continuous, and use 6) in "Additional material" to conclude that $c_g(m) = c_f(m)$ $\forall m$ and thus $f = g$ by 5) above.]

14. Find $f \in \mathcal{S}_1^{(2)}$ s.t. $-f''(x) + f(x) = \dfrac{1}{1 - \frac{1}{2} e^{2\pi i x}}$ $\forall x \in [0,1]$.

[Hint: Compute $(c_f(m))$].

15. Let $f : \mathbb{R} \to \mathbb{C}$ be the 1-periodic fcn s.t. $f(x) = x$, $0 \le x < 1$.

(Not continuous: ) Compute $(c_f(m))$

and calculate $\lim_{N\to\infty} \sigma_N(f)(0)$.

[Hint: Use "Additional material", 9)]

4. $\sin^2 y = -\frac{1}{4}\left(e^{2iy} + e^{-2iy} - 2\right) \quad \forall y \in \mathbb{R}$

$$\sin^2 \pi x \cdot F_N(x) = -\frac{1}{4}\left(e^{2\pi i x} + e^{-2\pi i x} - 2\right) \sum_{|n| \le N-1} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}$$

$$= \frac{1}{2} F_N(x) - \frac{1}{4} \sum_{n=-(N-1)}^{N-1} \left(1 - \frac{|n|}{N}\right) e^{2\pi i (n+1) x} - \frac{1}{4} \sum_{n=-(N-1)}^{N-1} \left(1 - \frac{|n|}{N}\right) e^{2\pi i (n-1) x}$$

$$= \frac{1}{2} \sum_{n=-(N-1)}^{N-1} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} - \frac{1}{4} \sum_{n=-(N-2)}^{N} \left(1 - \frac{|n-1|}{N}\right) e^{2\pi i n x} -$$

$$- \frac{1}{4} \sum_{n=-N}^{N-2} \left(1 - \frac{|n+1|}{N}\right) e^{2\pi i n x}$$

$$= -\frac{1}{4}\left(\left(1 - \frac{N-1}{N}\right) e^{2\pi i N x} + \left(1 - \frac{\overset{N-1}{\overset{\parallel}{|-N+1|}}}{N}\right) e^{-2\pi i N x} \right)$$

$$+ \sum_{n=-(N-1)}^{N-1} \underbrace{\left(\frac{1}{2}\left(\cancel{1} - \frac{|n|}{N}\right) - \frac{1}{4}\left(\cancel{1} - \frac{|n-1|}{N}\right) - \frac{1}{4}\left(\cancel{1} - \frac{|n+1|}{N}\right) \right)} e^{2\pi i n x}$$

$$= \frac{1}{4N}\left(|n+1| + |n-1| - 2|n| \right) = \frac{1}{4N} \begin{cases} 0 & n \ge 1 \\ 0 & n \le -1 \\ 2 & n = 0 \end{cases}$$

$$= \frac{1}{2N} - \frac{1}{4} \underbrace{\left(1 - \left(\frac{N-1}{N}\right)\right)}_{= \frac{1}{N}} \left(e^{2\pi i N x} + e^{-2\pi i N x}\right)$$

$$= \frac{1}{2N}\left(1 - \cos 2\pi N x\right) = \left\{ \text{Recall: } \sin^2 y = \frac{1 - \cos 2y}{2} \right\}$$

$$= \frac{1}{N} \sin^2 \pi N x \quad \rightsquigarrow \quad \boxed{F_N(x) = \frac{1}{N}\left(\frac{\sin \pi N x}{\sin \pi x}\right)^2}$$

---

5. If $c_f(n) = 0 \ \forall n \in \mathbb{Z}$, then by Fejér's Thm:

$$f(x) = \lim_N (F_N * f)(x) = \lim_N \sum_{|n| \le N-1} \left(1 - \frac{|n|}{N}\right) \underset{0 \ \forall n}{\underset{\parallel}{c_f(n)}} e^{2\pi i n x} = 0 \quad \forall x$$

---

6. Let $f \in \mathcal{C}_1$ and assume $\exists |m_j| \to \infty$ s.t. $|c_f(m_j)| \ge \delta \ \forall j$ and $\delta > 0$

By Fejér's Thm, $\exists p \in T_1$ s.t. $\|f - p\|_\infty < \delta/2$ and thus

$$|c_f(m_j) - c_p(m_j)| \le \int_0^1 |f(x) - p(x)|\, dx \le \|f - p\|_\infty < \delta/2$$

Since $p \in T_1$, $c_p(m_j) = 0$ for all large enough j

$\rightsquigarrow \ \forall j \gg 1 \quad \delta \le |c_f(m_j)| < \delta/2$ — Impossible!

7. Let $f, g \in \mathcal{R}_1$ and fix $\varepsilon > 0$. By Fejér's thm, $\exists p, q \in T_1$

s.t. $\|f-p\|_\infty < \varepsilon$ $\quad \|g-q\|_\infty < \varepsilon$

$\rightsquigarrow \int_0^1 f(nx)g(x)\,dx = \int_0^1 (f(nx)-p(nx))\,g(x)\,dx + \int_0^1 p(nx)\,q(x)\,dx$

$+ \int_0^1 p(nx)(g(x)-q(x))\,dx$,

$$\rightsquigarrow \left| \int_0^1 f(nx)g(x)\,dx - \int_0^1 p(nx)q(x)\,dx \right| \leq \|f-p\|_\infty \underbrace{\int_0^1 |g|\,dx}_{\leq \|g\|_\infty} + \|g-q\|_\infty \underbrace{\int |p(nx)|\,dx}_{\leq \|f\|_\infty + \|p-f\|_\infty}$$

$$\leq \varepsilon(\|g\|_\infty + \|f\|_\infty + \varepsilon)$$

$$\begin{cases} p(x) = \sum_{k=-N}^{N} a_k e^{2\pi i k x} \\ q(x) = \sum_{\ell=-M}^{M} b_\ell e^{2\pi i \ell x} \end{cases}$$

$$\rightsquigarrow \int_0^1 p(nx)\,q(x)\,dx = \sum_{k=-N}^{N}\sum_{\ell=-M}^{M} a_k b_\ell \underbrace{\int_0^1 e^{2\pi i(kn+\ell)x}\,dx}_{=0}$$

(If $n > \max(M,N)$, $|k| \leq N$, $|\ell| \leq M$, then $kn+\ell = 0 \iff k = \ell = 0$ $\Big\}$ if $kn+\ell \neq 0$)

$$= a_0 b_0 = \int p \cdot \int q = \int (f+p-f) \int (g+q-g)$$

$$= \int f \cdot \int g + \underbrace{\int f \cdot \int (q-g)}_{|\cdot| \leq \varepsilon \cdot \|f\|_\infty} + \underbrace{\int (p-f) \int g}_{|\cdot| \leq \varepsilon \cdot \|g\|_\infty} + \underbrace{\int (p-f) \int (q-g)}_{< \varepsilon^2}$$

$$\rightsquigarrow \left| \int_0^1 f(nx)g(x)\,dx - \int_0^1 f(x) \int_0^1 g(x) \right| \leq \varepsilon(\|g\|_\infty + \|f\|_\infty + \varepsilon) + \varepsilon(\|g\|_\infty + \|f\|_\infty) + \varepsilon^2$$

$\forall n > \max(M,N)$

Since $\varepsilon > 0$ arbitrary, $\to 0$ as $n \to \infty$.

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8 a) $c_f(m) = \int_0^1 x(1-x)e^{-2\pi i m x} = \int_0^1 x \cdot e^{-2\pi i m x} - \int_0^1 x^2 \cdot e^{-2\pi i m x}\,dx$

$= \{\text{s. integrals } 320, 321\} \underset{m \neq 0}{=} -\frac{1}{4\pi^2 m^2}\left((-2\pi i m - 1) - (-1)\right) -$

$-\frac{1}{8\pi^3 i m^3}\left((-4\pi^2 m^2 + 4\pi i m + 2) - 2\right) = \dots = -\frac{1}{2\pi^2 m^2}$

$c_f(0) = \int_0^1 x\,dx - \int_0^1 x^2\,dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \quad \rightsquigarrow \quad c_f(m) = \begin{cases} 1/6 & m = 0 \\ -\frac{1}{2\pi^2 m^2} & m \neq 0 \end{cases}$

b) By 13) in additional methods: $f(0) = 0 = \frac{1}{6} - \frac{1}{2\pi^2}\sum_{m \neq 0} \frac{1}{m^2}$

$\rightsquigarrow \frac{\pi^2}{3} = 2 \cdot \sum_{m=1}^{\infty} \frac{1}{m^2}$, $\quad f(\frac{1}{2}) = \frac{1}{4} = \frac{1}{6} - \frac{1}{2\pi^2}\sum_{m \neq 0} \frac{e^{i\pi m}}{m^2}$, $e^{i\pi m} = (-1)^m$

9. Recall $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$, $|t| < 1$. Use $t = \frac{1}{2}\cos 2\pi x$

$$\rightsquigarrow \quad f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos^n 2\pi x = \sum_{n=0}^{\infty} \frac{1}{2^n} \left(\frac{e^{2\pi i x} + e^{-2\pi i x}}{2} \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{4^n} \sum_{k=0}^{n} \binom{n}{k} e^{2\pi i (n-k-k)x}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{4^n} \binom{n}{k} e^{2\pi i (n-2k)x}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{\left(\sum_{\substack{n-2k=m \\ 0\le k\le n}} \frac{1}{4^n} \binom{n}{k} \right)}_{=\, c_f(m) \text{ by 6) in "Additional material"}} e^{2\pi i m x}$$

In particular, $c_f(1) = \sum_{\substack{n-2k=1 \\ 0\le k\le n}} \frac{1}{4^n}\binom{n}{k} = \sum_{k=0}^{\infty} \frac{1}{4^{2k+1}} \binom{2k+1}{k}$

---

10. $c_g(m) = c_h(m)\, c_f(m) \quad \forall m \in \mathbb{Z}$

$$g(x) = \sum_{m=0}^{\infty} \left(-\frac{1}{3}\right)^m \cdot e^{2\pi i m x} \quad \rightsquigarrow \quad c_g(m) = \begin{cases} \left(-\frac{1}{3}\right)^m & m \ge 0 \\ 0 & m < 0 \end{cases}$$

From exercise 8a) $c_h(m) = \begin{cases} 1/6 & m = 0 \\ -\frac{1}{2\pi^2 m^2} & m \neq 0 \end{cases}$

$$\Rightarrow \quad c_f(m) = \frac{c_g(m)}{c_h(m)} = \begin{cases} 6 \cdot c_g(0) & m = 0 \\ -2\pi^2 m^2 \cdot c_g(m) & m \neq 0 \end{cases}$$

Recall from 'Additonal material' 4) $c_{g'}(m) = 2\pi i m\, c_g(m)$

$$\rightsquigarrow \quad c_{g''}(m) = -4\pi^2 m^2 c_g(m) = 2 \cdot c_f(m) \quad \forall m \neq 0$$

Ansatz: $f(x) = 2g''(x) + b$, b constant

Then: $\begin{cases} c_f(0) = b \\ c_f(m) = 2c_{g''}(m) \end{cases}$ $\rightsquigarrow$ so set $b = 6\, c_g(0) = \underline{\underline{6}}.$

$$\rightsquigarrow \quad f(x) = 2g''(x) + 6 = \left\{ g'(x) = \frac{-2\pi i}{3(1+\frac{1}{3}e^{2\pi i x})^2}, \; g''(x) = \boxed{-\frac{4\pi^2}{9(1+\frac{1}{3}e^{2ix})^3}} \right.$$

11. If $f: [-1/2, 1/2) \to \mathbb{C}$ · odd :

$$c_f(-m) = \int_{-1/2}^{1/2} f(x)\, e^{+2\pi i m x}\, dx = \int_{-1/2}^{1/2} \underbrace{f(-x)}_{= -f(x)} e^{-2\pi i m x}\, dx = -c_f(m) \quad \forall m$$

(in particular: $c_f(0) = 0$

· real-valued :

$$\overline{c_f(m)} = \int_{-1/2}^{1/2} f(x)\, e^{2\pi i m x}\, dx = c_f(-m)$$

· If $\sum_{\mathbb{Z}} |c_f(m)| < +\infty$, so that $f(x) = \sum_{\mathbb{Z}} c_f(m)\, e^{2\pi i m x} \quad \forall x$

and f is odd + real-valued :

$$f(x) = \sum_{m=-\infty}^{-1} c_f(m)\, e^{2\pi i m x} + \sum_{m=1}^{\infty} c_f(m)\, e^{2\pi i m x}$$

$$= \sum_{m=1}^{\infty} \left(c_f(-m)\, e^{-2\pi i m x} + c_f(m)\, e^{2\pi i m x} \right)$$

to be determined.

$$= \sum_{m=1}^{\infty} \underbrace{2i\, c_f(m)}_{= b_f(m)} \underbrace{\left(\frac{e^{2\pi i m x} - e^{-2\pi i m x}}{2i} \right)}_{= \sin 2\pi m x} = \sum_{m=1}^{\infty} b_f(m) \sin 2\pi m x$$

Note : $2i \cdot c_f(m) = -2i \cdot \overline{c_f(-m)} = \overline{2i \cdot c_f(m)} \leadsto 2i \cdot c_f(m)$ is real.

f real-valued $\qquad f$ odd

$$2i \cdot c_f(m) = 2i \int_{-1/2}^{1/2} f(x)\, e^{-2\pi i m x}\, dx = 2i \underbrace{\int_{-1/2}^{1/2} \underbrace{f(x) \cos 2\pi m x}_{\text{odd fcn}}\, dx}_{= 0}$$

$$+ \, 2 \cdot \underbrace{\int_{-1/2}^{1/2} \underbrace{f(x) \sin 2\pi m x}_{\text{even fcn}}\, dx}_{= 2 \int_0^{1/2} f(x) \sin 2\pi m x\, dx}$$

$$\leadsto \quad b_f(m) = 4 \cdot \int_0^{1/2} f(x) \sin 2\pi m x\, dx \qquad (\text{"sinus coefficients"}) \quad m \geq 1.$$

In our case : $f(x) = x\left(\frac{1}{2} - x\right) \quad 0 \leq x < 1/2$

$$b_f(m) = 2 \int_0^{1/2} x \sin 2\pi m x\, dx - 4 \int_0^{1/2} x^2 \sin 2\pi m x\, dx = \{\text{p-ints : } 216, 218\}$$

$$= 2 \cdot \frac{1}{(2\pi m)^2} \left[\sin 2\pi m x - 2\pi m \cdot x \cdot \cos 2\pi m x \right]_0^{1/2} -$$

$$- 4 \cdot \frac{1}{(2\pi m)^3} \left[2 \cos 2\pi m x + 2 \cdot 2\pi m \cdot x \cdot \sin 2\pi m x - (2\pi m)^2 \cdot x^2 \cdot \cos 2\pi m x \right]_0^{1/2}$$

$$= \frac{1}{2\pi^2 m^2} \left(\cancel{-\pi m (-1)^m} \right) - \frac{1}{2\pi^3 m^3} \left(2(-1)^m + \cancel{\frac{\pi^2 m^2}{2}(-1)^m} - 2 \right)$$

$$= \frac{1}{\pi^3 m^3} \left(1 - (-1)^m \right) = \begin{cases} 0 & m = 2k \\ \frac{2}{\pi^3 (2k-1)^3} & m = 2k-1 \end{cases}, \; k \geq 1 \quad \leadsto \quad f(x) = \frac{2}{\pi^3} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \sin((2k-1)x)$$

12/13 Since $c_{f''}(m) = -4\pi^2 m^2 c_f(m)$ and $|c_{f''}(m)| \le \|f''\|_\infty \ \forall m$

$$\sum_{m=-\infty}^{\infty} |c_f(m)| \le \|f''\|_\infty + \frac{1}{4\pi^2} \sum_{m \neq 0} \frac{1}{m^2} < +\infty$$

$\rightsquigarrow$ $\underline{g}(x) = \sum_{m=-\infty}^{\infty} c_f(m) e^{2\pi i m x}$ is cnts by Weierstraß M-test

and thus $c_g(m) = c_f(m)$ by "6) in "additional material"

$\rightsquigarrow$ $c_{g-f}(m) = 0$ by linearity and thus $g - f \equiv 0$ by Exercise 5), i.e. $f = g$.

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14. $g(x) = \dfrac{1}{1 - \frac{1}{2} e^{2\pi i x}} = \sum_{m=0}^{\infty} \dfrac{1}{2^m} e^{2\pi i m x}$ $\rightsquigarrow$ $c_g(m) = \begin{cases} 1/2^m & m \ge 0 \\ 0 & m < 0 \end{cases}$

$$-f'' + f = g \Rightarrow \underbrace{-c_{f''}(m) + c_f(m)}_{= (4\pi^2 m^2 + 1) c_f(m)} = c_g(m) \ \forall m$$

$$\Rightarrow c_f(m) = \frac{c_g(m)}{1 + 4\pi^2 m^2} = \begin{cases} 0 & m < 0 \\ \dfrac{1}{2^m (1 + 4\pi^2 m^2)} & m \ge 0 \end{cases}$$

$$\Rightarrow \boxed{f(x) = \sum_{m=0}^{\infty} \frac{1}{2^m (1 + 4\pi^2 m^2)} e^{2\pi i m x}} \quad \forall x \in [0,1]$$

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15. $c_f(m) = \int_0^1 x \cdot e^{-2\pi i x m} \, dx \underset{m \neq 0}{=} \left[\dfrac{e^{-2\pi i m x}}{(-2\pi i m)^2} (-2\pi i m \cdot x - 1) \right]_0^1$

$$= -\frac{1}{4\pi^2 m^2} (-2\pi i m - 1) + \frac{1}{4\pi^2 m^2} (-1) = \frac{i}{2\pi m}$$

$c_f(0) = \int_0^1 x \, dx = \frac{1}{2}$ $\rightsquigarrow$ $c_f(m) = \begin{cases} 1/2 & m = 0 \\ \dfrac{i}{2\pi m} & m \neq 0 \end{cases}$

By "Additional" 9) $(\sigma_N * f)(0) = \frac{1}{2} \big(\underset{0}{\underset{\parallel}{f(0^+)}} + \underset{1}{\underset{\parallel}{f(0^-)}} \big) = \underline{\underline{\frac{1}{2}}}$
