Main results, MMG710

Fourier series:

$f : \mathbb{R} \to \mathbb{C}$ Riemann-integrable 1-periodic fcn.

$c_f(m) = \int_I f(x) e^{-2\pi i m x}\, dx$, $m \in \mathbb{Z}$, I = any interval of length 1.

Fejer kernel: $F_N(x) = \sum_{|m| \le N} \left(1 - \frac{|m|}{N}\right) e^{2\pi i m x} = \frac{1}{N} \left(\frac{\sin \pi N x}{\sin \pi x} \right)^2 \quad \forall N \geq 1.$

Main properties:
i) $F_N(x) \geq 0$, $\int_I F_N(x)\, dx = 1 \quad \forall N$

ii) F_N even.

iii) $\forall \delta > 0 \quad \int_{\delta \le |x| < 1/2} F_N(x)\, dx \to 0$, as $N \to \infty$.

Obs: $(F_N * f)(x) = \int_I F_N(y) f(x-y)\, dy = \sum_{|m| \le N} \left(1 - \frac{|m|}{N}\right) c_f(m) e^{2\pi i m x} \quad \forall x \in \mathbb{R}$

Thm (Fejer) i) f piecewise cnts; $\forall x \in \mathbb{R}$ $\lim_{y \to x^+} f(y) = f(x^+)$ and $\lim_{y \to x^-} f(y) =: f(x^-)$ exist. $\Rightarrow$ $(F_N * f)(x) \to \frac{1}{2}\left(f(x^+) + f(x^-)\right) \quad \forall x$

ii) f continuous. $\Rightarrow$ $F_N * f \to f$ uniformly.

Proof. i) Fix $\varepsilon > 0$ and $x \in \mathbb{R}$. By assumption, $\exists \delta > 0$ s.t.

- $|f(x^+) - f(x-y)| < \varepsilon \quad \forall -\delta < y < 0$
- $|f(x^-) - f(x-y)| < \varepsilon \quad \forall\ 0 < y < \delta$

(If x is a continuity pt of f, there is no reason to split into two cases)

Since F_N is even and $\int_{-1/2}^{1/2} F_N(y)\, dy = 1$,

we have $\int_{-1/2}^{0} F_N\, dy = \int_0^{1/2} F_N\, dy = 1/2$. $\Rightarrow$ $(F_N * f)(x) - \frac{1}{2}\left(f(x^+) + f(x^-)\right)$

$$= \int_{-1/2}^{0} F_N(y) \left(f(x-y) - f(x^+)\right) dy + \int_0^{1/2} F_N(y) \left(f(x-y) - f(x^-)\right) dy$$

$$= \int_{-1/2}^{-\delta} F_N(y) \big(\underbrace{f(x-y) - f(x^+)}_{\le 2\|f\|_\infty}\big) dy + \int_{-\delta}^{0} F_N(y) \big(\underbrace{f(x-y) - f(x^+)}_{|\cdot| < \varepsilon}\big) dy$$

$$+ \int_0^{\delta} F_N(y) \big(\underbrace{f(x-y) - f(x^-)}_{|\cdot| < \varepsilon}\big) dy + \int_\delta^{1/2} F_N(y) \big(\underbrace{f(x-y) - f(x^-)}_{\le 2\|f\|_\infty}\big) dy$$

$$\Rightarrow \left| (F_N * f)(x) - \frac{1}{2}\left(f(x^+) + f(x^-)\right) \right| \le \varepsilon + 4\|f\|_\infty \cdot \int_\delta^{1/2} F_N(y)\, dy$$

$$\Rightarrow \overline{\lim_N} \left| \quad \right| \le \varepsilon \quad (\varepsilon > 0 \text{ arbitrary})$$

ii) f uniformly cnts: Fix $\varepsilon > 0$ and pick $\delta > 0$ s.t. $|y| < \delta \Rightarrow |f(x) - f(x-y)| < \varepsilon \ \forall x$

$$\Rightarrow (F_N * f)(x) - f(x) = \int_{-1/2}^{1/2} F_N(y) \left(f(x-y) - f(y)\right) dy = \int_{|y| < \delta} F_N(y) \big(\underbrace{f(x-y) - f(y)}_{< \varepsilon}\big) dy$$

$$+ \int_{\delta \le |y| < 1/2} F_N(y) \big(\underbrace{f(x-y) - f(y)}_{\le 2\|f\|_\infty}\big) dy \Rightarrow \overline{\lim_N} \left|(F_N * f)(x) - f(x)\right| \le \varepsilon \quad (\varepsilon \text{ arbitrary})$$

Cor. f_1, f_2 continuous, $c_{f_1}(m) = c_{f_2}(m)\ \forall m \in \mathbb{Z} \Rightarrow f_1 = f_2$.

Proof. $f_1(x) - f_2(x) \overset{\text{Fejer.}}{=} \lim_N (F_N * (f_1 - f_2))(x) = \lim_N \sum_{|m| \leq N} \left(1 - \frac{|m|}{N}\right) \underbrace{(c_{f_1}(m) - c_{f_2}(m))}_{=0} e^{2\pi i m x}$

$= 0 \quad \forall x \in \mathbb{R}$ $\quad\times$

Cor. $\sum_{m=-\infty}^{\infty} |c_f(m)| < +\infty \Rightarrow f(x) = \sum_{m=-\infty}^{\infty} c_f(m) e^{2\pi i m x} \quad \forall x \in \mathbb{R}$

Proof. Set $g(x) = \sum_{m=-\infty}^{\infty} c_f(m) e^{2\pi i m x}$ (exists $\forall x \in \mathbb{R}$ + continuous)

$c_g(m) = c_f(m)$ (by unif. convergence of the sum) $\Rightarrow f = g$. $\quad\times$

Inner product : $f, g : \mathbb{R} \to \mathbb{C}$ Riemann integrable (1-periodic), $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$

Note: • $\|f\|^2 = \langle f, f\rangle \geq 0$

• $\|f - g\|^2 = \|f\|^2 + \|g\|^2 - 2\,\mathrm{Re}\,\langle f, g\rangle$

• $e_m(x) = e^{2\pi i m x}$, $m \in \mathbb{Z}$ – $\|e_m\| = 1$ and $\langle e_m, e_n\rangle = 0\ \forall m \neq n$

$$\boxed{\begin{array}{l}\text{Bessel's inequality: } \forall f : \mathbb{R} \to \mathbb{C} \text{ Riemann-integrable 1-periodic fct.} \\ \displaystyle\sum_{m=-\infty}^{\infty} |c_f(m)|^2 \leq \int_0^1 |f(x)|^2\,dx\end{array}}$$

Proof. $\forall N \geq 1 \quad \left\| f - \sum_{m=-N}^{N} \underbrace{\langle f, e_m\rangle}_{=c_f(m)} e_m \right\|^2 \geq 0$

$= \|f\|^2 - \underbrace{\left\| \sum_{m=-N}^{N} c_f(m) e_m \right\|^2}_{= \sum_{m=-N}^{N} |c_f(m)|^2} - 2\,\mathrm{Re}\,\underbrace{\left\langle f, \sum_{m=-N}^{N} \langle f, e_m\rangle e_m \right\rangle}_{\sum_{m=-N}^{N} \underbrace{\overline{\langle f, e_m\rangle} \cdot \langle f, e_m\rangle}_{= |c_f(m)|^2 \geq 0}}$

$= \|f\|^2 - \sum_{m=-N}^{N} |c_f(m)|^2$

$\Rightarrow \sum_{|m| \leq N} |c_f(m)|^2 \leq \|f\|^2 \quad \forall N \rightsquigarrow \sum_{m=-\infty}^{\infty} |c_f(m)|^2 \leq \int_0^1 |f(x)|^2\,dx$ $\quad\times$

Cor. Sps. f' exists and is continuous. Then $\sum_{m=-\infty}^{\infty} |c_f(m)| < +\infty$.

Proof. $c_{f'}(m) = 2\pi i m\, c_f(m)\ \forall m \Rightarrow \sum_{m \neq 0} |c_f(m)| = \sum_{m \neq 0} \frac{|c_{f'}(m)|}{2\pi |m|}$

$\overset{(CS)}{\leq} \underbrace{\left(\sum_{m \neq 0} \frac{1}{4\pi^2 m^2}\right)^{1/2}}_{< +\infty} \cdot \underbrace{\left(\sum_{m \neq 0} |c_{f'}(m)|^2\right)^{1/2}}_{\leq \left(\int_0^1 |f'(x)|^2\,dx\right)^{1/2} < +\infty}$ (since f' is continuous)

$\times$

Cor. (Parseval's identity) $\forall f: \mathbb{R} \to \mathbb{C}$ Riemann-int. + 1-periodic (and $\int_0^1 |f(x)|^2 dx < +\infty$)

$$\sum_{m=-\infty}^{\infty} |c_f(m)|^2 = \int_0^1 |f(x)|^2 dx$$

Proof. Set $f^*(x) = \overline{f(-x)}$, and note that $c_{f^*}(m) = \overline{c_f(m)}$ $\forall m \in \mathbb{Z}$

Let $g = f * f^*$ so that $c_g(m) = |c_f(m)|^2$ $\forall m \in \mathbb{Z}$.

By Bessel: $\sum_{m=-\infty}^{\infty} |c_g(m)| \leq \int_0^1 |f(x)|^2 dx < +\infty$

$\Rightarrow g(x) = \sum_{m=-\infty}^{\infty} |c_f(m)|^2 e^{2\pi i m x}$ $\forall x \in \mathbb{R} \Rightarrow g(0) = \int_0^1 |f(x)|^2 dx = \sum_{m=-\infty}^{\infty} |c_f(m)|^2$ $\boxtimes$

Fourier transforms:

$f: \mathbb{R} \to \mathbb{C}$ Riemann-integrable, $\int_{-\infty}^{\infty} |f(x)| dx < +\infty$

$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$, $\xi \in \mathbb{R}$.

Fejer kernel: $F_T(x) = \int_{-T}^{T} (1 - |\xi|/T) e^{2\pi i \xi x} d\xi = \frac{1}{T}\left(\frac{\sin \pi T x}{\pi x}\right)^2$

Main properties: i) $F_T \geq 0$ and $\int_{-\infty}^{\infty} F_T(x) dx = 1$ ← check notes (Module IV)

ii) F_T even.

iii) $\forall \delta > 0$ $\int_{|y| \geq \delta} F_T(y) dy \to 0$, as $T \to \infty$.

Obs: $(F_T * f)(x) = \int_{-T}^{T} \left(1 - \frac{|\xi|}{T}\right) \hat{f}(\xi) e^{2\pi i \xi x} d\xi$ $\forall x \in \mathbb{R}$.

Thm (Fejer) i) f piecewise continuous; $\lim_{y \to x^+} f(y) = f(x^+)$ and $\lim_{y \to x^-} f(y) = f(x^-)$ exist $\forall x \Rightarrow (F_T * f)(x) \to \frac{1}{2}(f(x^+) + f(x^-))$

ii) f continuous $\Rightarrow F_T * f \to f$ uniformly.

(**Proof.** As in Fourier series-part)

Cor. 1 $f_1, f_2: \mathbb{R} \to \mathbb{C}$ continuous, $\hat{f}_1 = \hat{f}_2 \Rightarrow f_1 = f_2$

Cor. 2 If $\int_{-\infty}^{\infty} |\hat{f}(\xi)| d\xi < +\infty$, then $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$

Proof. Set $g(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$, $x \in \mathbb{R}$

By Fejer, $f(x) = \lim_{T} \int_{-T}^{T} (1 - |\xi|/T) \hat{f}(\xi) e^{2\pi i \xi x} d\xi$

$$|g(x) - f(x)| = \lim_T \left| \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi - \int_{-T}^{T} (1 - |\xi|/T) \hat{f}(\xi) e^{2\pi i \xi x} d\xi \right|$$

$$= \overline{\lim_T} \Bigg| \underbrace{\int_{|\xi|>T} \hat{f}(\xi) e^{2\pi i \xi x} d\xi}_{\leq \underbrace{\int_{|\xi|>T} |\hat{f}(\xi)| d\xi}_{\to 0,\ T \to \infty}} + \underbrace{\frac{1}{T} \int_{-T}^{T} |\xi| \hat{f}(\xi) e^{2\pi i \xi x} d\xi}_{\leq \underbrace{\frac{1}{T}\int_{-\varepsilon T}^{\varepsilon T} |\xi| |\hat{f}(\xi)|}_{< \varepsilon \cdot \int_{-\infty}^{\infty} |\hat{f}(\xi)| d\xi} + \underbrace{\int_{|\xi| \geq \varepsilon T} |\hat{f}(\xi)| d\xi}_{\to 0,\ \text{as } T \to \infty}} \Bigg| < \varepsilon \cdot \int_{-\infty}^{\infty} |\hat{f}(\xi)| d\xi$$

fix $\varepsilon > 0$

ε arbitrary!

$\boxtimes$

Thm (Plancherel) $\forall f:\mathbb{R}\to\mathbb{C}$ Riemann int., $\int|f|, \int|f|^2<+\infty$.

$$\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|\hat{f}(\xi)|^2 d\xi$$

Proof. Set $f^*(x)=\overline{f(-x)}$; note that $\widehat{f^*}(\xi)=\overline{\hat{f}(\xi)}$ $\forall \xi\in\mathbb{R}$.

If $g=f*f^*$, then $\hat{g}(\xi)=|\hat{f}(\xi)|^2$ $\forall\xi\in\mathbb{R}$.

By Fejer's thm: $g(0)=\lim_T \int_{-T}^{T}\left(1-\frac{|\xi|}{T}\right)|\hat{f}(\xi)|^2d\xi \leq \int_{-\infty}^{\infty}|\hat{f}(\xi)|^2d\xi$

$\qquad\qquad \int|f(x)|^2dx = g(0)$

$\qquad\qquad \geq \leftarrow$ fix $\varepsilon>0$

$$\lim_T \int_{-\varepsilon T}^{\varepsilon T}\left(1-\frac{|\xi|}{T}\right)|\hat{f}(\xi)|^2d\xi \geq (1-\varepsilon)\int_{-\infty}^{\infty}|\hat{f}(\xi)|^2d\xi$$

$$\Rightarrow (1-\varepsilon)\int_{-\infty}^{\infty}|\hat{f}(\xi)|^2d\xi \leq \int_{-\infty}^{\infty}|f(x)|^2dx \leq \int_{-\infty}^{\infty}|\hat{f}(\xi)|^2d\xi \quad \forall\varepsilon>0 \quad \boxtimes.$$

Cor. Sps $\int_{-\infty}^{\infty}|f'|\,dx<+\infty$, $\int_{-\infty}^{\infty}|f'|^2dx<+\infty$, then $\int_{-\infty}^{\infty}|\hat{f}(\xi)|\,d\xi<+\infty$.

Proof.

$$\int_{-\infty}^{\infty}|\hat{f}(\xi)|\,d\xi \leq \underbrace{\left|\int_{-\infty}^{\infty}f(x)dx\right|}_{<+\infty} + \underbrace{\int_{|\xi|\geq 1/2}\frac{|\widehat{f'}(\xi)|}{2\pi|\xi|}d\xi}$$

$\widehat{f'}(\xi)=2\pi i\xi\hat{f}(\xi)$

$$\overset{CS}{\leq} \underbrace{\left(\int_{|\xi|\geq\frac12}\left(\frac{1}{2\pi|\xi|}\right)^2d\xi\right)^{1/2}}_{<+\infty}\underbrace{\left(\int_{|\xi|\geq 1/2}|\widehat{f'}(\xi)|^2d\xi\right)^{1/2}}$$

$$\underset{\text{Plancherel}}{\leq} \int_{-\infty}^{\infty}\underbrace{|f'|^2dx}_{<+\infty}.$$