MODULE IV

- Central Limit Theorem (CLT)
- Plancherel's Theorem

<u>Aim</u>: Break down the proof of CLT to Fourier analysis — Fejer's thm + Plancherel's thm.

Important to focus on:

- Fejer's thm (In particular $\int_{-\infty}^{\infty} |\hat{f}| < +\infty \Rightarrow f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$)
- Plancherel's thm
- Explicit examples ($f(x) = e^{-\pi x^2}$, $f(x) = e^{-2\pi |x|}$, ...)

Central Limit Theorem

Toss a (fair) coin many times and record the outcomes:

set $\theta_n = \begin{cases} +1 & \text{if } n\text{th toss} = \text{head} \\ -1 & \quad \text{---}\,''\,\text{---} \quad = \text{tail} \end{cases}, \quad n \geq 1.$

Intuition tells us that $\dfrac{\theta_1 + \dots + \theta_N}{N} \approx 0$ for "most" sequences $(\theta_1, \dots, \theta_N)$ of ± 1 if N is large.

The Central Limit theorem (CLT) quantifies this intuition and provides an explicit "renormalization". Our aim this week will be to prove the following version of the CLT:

set $\{-1,1\}^N = \{(\theta_1, \dots, \theta_N) : \theta_n \in \{-1,1\}, n = 1, \dots, N\}$ $\quad (\# \{-1,1\}^N = 2^N)$

$$\boxed{\textbf{Thm (CLT)} \quad \forall a < b: \quad \frac{\# \left\{ (\theta_1, \dots, \theta_N) \in \{-1,1\}^N : a \leq \frac{\theta_1 + \dots + \theta_N}{\sqrt{N}} \leq b \right\}}{2^N} \xrightarrow[N \to \infty]{} \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx}$$

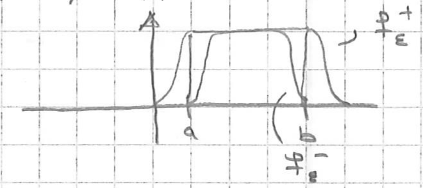
Strategy of proof: set $I = [a,b]$; we wish to prove

$$\frac{1}{2^N} \sum_{\underline{\theta} \in \{-1,1\}^N} \chi_I \left(\frac{\theta_1 + \dots + \theta_N}{\sqrt{N}} \right) \xrightarrow[N \to \infty]{} \frac{1}{\sqrt{2\pi}} \int_I e^{-x^2/2}\, dx \qquad (*)$$

Step 1: $\forall \varepsilon > 0 \;\; \exists f_\varepsilon^\pm \in \mathcal{C}^\infty_{cpt}(\mathbb{R})$ (compactly supported + smooth)

s.t. i) $f_\varepsilon^- \leq \chi_I \leq f_\varepsilon^+$

Prove later! ii) $\int_{-\infty}^{\infty} (f_\varepsilon^+ - f_\varepsilon^-)(x)\, dx < \varepsilon$



Step 2: $\forall f \in \mathcal{C}^2_{cpt}(\mathbb{R})$:

$$\frac{1}{2^N} \sum_{\underline{\theta} \in \{-1,1\}^N} f \left(\frac{\theta_1 + \dots + \theta_N}{\sqrt{N}} \right) \xrightarrow[N \to \infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-x^2/2}\, dx$$

Proof of CLT assuming step 1 + 2:

(Step 1) $\forall \varepsilon > 0$: $\quad \dfrac{1}{2^N} \displaystyle\sum_{\underline{\theta} \in \{\pm 1\}^N} f_\varepsilon^- \left(\frac{\theta_1 + \dots + \theta_N}{\sqrt{N}} \right) \leq$ LHS of $*$ $\leq \dfrac{1}{2^N} \displaystyle\sum_{\underline{\theta} \in \{\pm 1\}^N} f_\varepsilon^+ \left(\frac{\theta_1 + \dots + \theta_N}{\sqrt{N}} \right)$

$\downarrow$ Step 2 $\qquad\qquad \downarrow$ Step 2

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_\varepsilon^-(x)\, e^{-x^2/2}\, dx \qquad\qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_\varepsilon^+(x)\, e^{-x^2/2}\, dx$$

$$\rightsquigarrow \underline{\lim_N} \text{ LHS of } (*) \geq \frac{1}{\sqrt{2\pi}} \int_I e^{-x^2/2}\, dx - \underbrace{\int_{-\infty}^{\infty} \left(f_\varepsilon^+(x) - f_\varepsilon^-(x)\right) \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx}_{\leq \frac{\varepsilon}{\sqrt{2\pi}}}$$

$$\overline{\lim_N} \text{ LHS of } (*) \leq \frac{1}{2\pi} \int_I e^{-x^2/2}\, dx + \underbrace{\int_{-\infty}^{\infty} \left(f_\varepsilon^+(x) - f_\varepsilon^-(x)\right) e^{-x^2/2}\, dx}_{\leq \varepsilon/2\pi}$$

since $\varepsilon > 0$ arbitrary, we are done!

Proof of step 2

If $f \in C(\mathbb{R})$ and $\int_{-\infty}^{\infty} |f(x)| dx < +\infty$, define its <u>Fourier transform</u> $\hat{f}$ by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx, \quad \xi \in \mathbb{R}.$$

<u>Proposition</u>:

(i) If $\int_{-\infty}^{\infty} |\hat{f}(\xi)| d\xi < +\infty$, then $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \quad \forall x \in \mathbb{R}$

(ii) If $f \in C^2_{cpt}(\mathbb{R})$, then $|\hat{f}(\xi)| \leq \frac{c}{1+|\xi|^2} \quad \forall \xi \in \mathbb{R}$ (for some constant c)

(Prove later!)

<u>Proof of step 2</u>: Let $f \in C^2_{cpt}(\mathbb{R})$ so that by Proposition:

- $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \quad \forall x \in \mathbb{R}$
- $|\hat{f}(\xi)| \leq \frac{c}{1+|\xi|^2}$

$$\rightsquigarrow \frac{1}{2^N} \sum_{\underline{\theta} \in \{-1,1\}^N} f\left(\frac{\theta_1 + \dots + \theta_N}{\sqrt{N}}\right) = \int_{-\infty}^{\infty} \hat{f}(\xi) \left(\frac{1}{2^N} \sum_{\underline{\theta} \in \{-1,1\}^N} e^{2\pi i \xi \left(\frac{\theta_1 + \dots + \theta_N}{\sqrt{N}}\right)} \right) d\xi$$

$$\frac{1}{2^N} \sum_{\underline{\theta} \in \{-1,1\}^N} e^{2\pi i \xi \left(\frac{\theta_1 + \dots + \theta_N}{\sqrt{N}}\right)} = \prod_{n=1}^{N} \frac{1}{2}\left(e^{\frac{2\pi i \xi}{\sqrt{N}}} + e^{-\frac{2\pi i \xi}{\sqrt{N}}}\right) = \cos^N\left(\frac{2\pi\xi}{\sqrt{N}}\right)$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi) \cos^N\left(\frac{2\pi\xi}{\sqrt{N}}\right) d\xi$$

<u>Obs</u>: $\cos^N\left(\frac{2\pi\xi}{\sqrt{N}}\right) \underset{\substack{\uparrow \\ \text{Taylor} \\ \exp \approx 0}}{=} \left(1 - \frac{2\pi^2 \xi^2}{N} + O(N)\right)^N \longrightarrow e^{-2\pi^2\xi^2} \quad \forall \xi \in \mathbb{R} \quad (**)$

(<u>Recall</u>: $\left(1 + \frac{\alpha_n}{n}\right)^n \to e^{\alpha}$ if $\alpha = \lim_n \alpha_n$.)

<u>Exercise</u>: Show that $\forall R > 0$:

$$\lim_{N \to \infty} \sup_{|\xi| \leq R} \left| \cos^N\left(\frac{2\pi\xi}{\sqrt{N}}\right) - e^{-2\pi^2\xi^2} \right| = 0 \quad (***)$$

$\forall R > 0$

$$= \underbrace{\int_{-R}^{R} \hat{f}(\xi) \cos^N\left(\frac{2\pi\xi}{\sqrt{N}}\right) d\xi}_{\to \hat{f}(\xi) e^{-2\pi^2\xi^2} d\xi \text{ uniformly!}} + \underbrace{\int_{|\xi| > R} \hat{f}(\xi) \overbrace{\cos^N\left(\frac{2\pi\xi}{\sqrt{N}}\right)}^{|\cdot| \leq 1} d\xi}_{|\cdot| \leq \int_{|\xi|>R} |\hat{f}(\xi)| d\xi \leq c'/R}$$

for some c'; $|\hat{f}(\xi)| \leq \frac{c}{1+|\xi|^2}$

$$\longrightarrow \int_{-R}^{R} \hat{f}(\xi) e^{-2\pi^2\xi^2} d\xi$$

$$\underset{R \to \infty}{\longrightarrow} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-2\pi^2\xi^2} d\xi + 0$$

Thm (Plancherel) $\forall f, g \in \mathcal{C}(\mathbb{R})$ with $\int_{-\infty}^{\infty} |f(x)|\,dx,\ \int_{-\infty}^{\infty} |g(x)|\,dx < +\infty$ ($\int_{-\infty}^{\infty} |f(x)|^2,\ \int |g(x)|^2 dx < +\infty$)

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi.$$

Lemma If $h(x) = e^{-\pi x^2}$, then $\hat{h}(\xi) = e^{-\pi \xi^2}$

Obs: If $g_\alpha(x) = e^{-\alpha x^2}$, $\alpha > 0$, then

$$\hat{g}_\alpha(\xi) = \int_{-\infty}^{\infty} e^{-\pi\left(\frac{\sqrt{\alpha}}{\sqrt{\pi}}x\right)^2} \cdot e^{-2\pi i \left(\frac{\sqrt{\pi}}{\sqrt{\alpha}}\right)\xi \cdot \left(\frac{\sqrt{\alpha}}{\sqrt{\pi}}x\right)}\,dx = \left\{ y = \frac{\sqrt{\alpha}}{\sqrt{\pi}}x \quad dy = \frac{\sqrt{\alpha}}{\sqrt{\pi}}dx \right\}$$

$$= \frac{\sqrt{\pi}}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-\pi y^2} \cdot e^{-2\pi i \left(\frac{\sqrt{\pi}}{\sqrt{\alpha}}\xi\right) y}\,dy = \frac{\sqrt{\pi}}{\sqrt{\alpha}}\,\hat{h}\left(\frac{\sqrt{\pi}}{\sqrt{\alpha}}\xi\right)$$

$$= \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\pi^2 \xi^2}{\alpha}} \quad \forall \xi \quad \text{(by Lemma)}.$$

$$\leadsto \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-2\pi^2\xi^2}\,d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}_{1/2}(\xi)}\,d\xi$$

$$= \{\text{Plancherel}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-x^2/2}\,dx$$ (finishes the proof of step 2)

Remains to prove:

- Step 1 (Smoothing) — See exercise
- Plancherel's Thm.
- Lemma + Proposition above

General properties of Fourier transform.

Proof of Lemma: $\hat{h}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{-2\pi i x \xi}\,dx = e^{-\pi\xi^2} \underbrace{\int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\,dx}_{= F(\xi)}$

Want to show: $F(\xi) = 1 \quad \forall \xi$

$$F'(\xi) = -2\pi i \int_{-\infty}^{\infty} (x + i\xi) e^{-\pi(x+i\xi)^2}\,dx$$ (convince yourself that interchanging diff/int. is allowed)

$$= -i \int_{-\infty}^{\infty} \frac{d}{dx} e^{-\pi(x+i\xi)^2}\,dx = 0 \quad \leadsto \quad F(\xi) = c \quad \forall \xi \in \mathbb{R}.$$

$$\leadsto c = F(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1 \quad \text{(exercise!)} \quad \leadsto \quad \hat{h}(\xi) = e^{-\pi\xi^2}$$

$$\left(\left(\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)}\,dx\,dy = \{\text{polar coord.}\} = 2\pi \int_0^{\infty} r e^{-\pi r^2}\,dr \right.$$

$$\left. = 2\pi \left[-\frac{1}{2\pi} e^{-\pi r^2}\right]_0^{\infty} = 1. \right)$$

General properties

Fejér's kernel: $F_T(x) = \int_{-T}^{T} (1 - |\xi|/T)\, e^{2\pi i \xi x}\, d\xi = \begin{cases} T & \text{if } x = 0 \\ \frac{1}{T}\left(\frac{\sin \pi T x}{\pi x}\right)^2 & x \neq 0 \end{cases}$

exercise!

(Cp. with Fourier series:

$$F_N(x) = \begin{cases} N & \text{if } x = 0 \\ \frac{1}{N}\left(\frac{\sin \pi N x}{\sin \pi x}\right)^2 & x \neq 0 \end{cases} \quad)$$

Note: $(F_T * f)(x) = \int_{-\infty}^{\infty} F_T(y) f(x-y)\, dy =$

$$= \int_{-\infty}^{\infty} \int_{-T}^{T} (1 - |\xi|/T)\, \underbrace{e^{2\pi i \xi y}}_{e^{2\pi i \xi (y-x)} \cdot e^{2\pi i \xi x}}\, f(x-y)\, d\xi\, dy$$

$$= \int_{-T}^{T} \int_{-\infty}^{\infty} (1 - |\xi|/T)\, e^{2\pi i \xi x} \cdot f(x-y)\, e^{-2\pi i \xi (x-y)}\, dy\, d\xi$$

(Verify legality of changing order of integ.)

$$= \int_{-T}^{T} (1 - |\xi|/T)\, \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi$$

if $\int_{-\infty}^{\infty} |f(x)|\, dx < +\infty$.

(Fejér's thm)

Lemma If f is continuous at x, then $(F_T * f)(x) \to f(x)$, as $T \to \infty$.

(If f is uniformly continuous on $\mathbb{R}$, then $F_T * f \to f$ uniformly)

Proof:

Need: $\int_{-\infty}^{\infty} F_T(x)\, dx = 1 \quad \forall T > 0$ (see proof below)

$\leadsto (F_T * f)(x) - f(x) = \int_{-\infty}^{\infty} F_T(y)\left(f(x-y) - f(x)\right) dy =$ { if f is unif. continuous; pick $\varepsilon > 0$ and δ s.t. $|s-t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon$ }

$$= \int_{-\delta}^{\delta} F_T(y) \underbrace{\left(f(x-y) - f(y)\right)}_{|\cdot| < \varepsilon} dy + \int_{|y| \geq \delta} F_T(y) \underbrace{\left(f(x-y) - f(y)\right)}_{|\cdot| \leq 2\|f\|_\infty} dy$$

$$\leq \varepsilon \underbrace{\int_{-\infty}^{\infty} F_T(y)\, dy}_{=1} + 2\|f\|_\infty \underbrace{\int_{|y| \geq \delta} \frac{1}{T} \underbrace{\left(\frac{\sin \pi T y}{\pi y}\right)^2}_{\leq \frac{1}{\pi^2}\frac{1}{y^2}} dy}_{} =$$

$$\leq \frac{1}{T} \frac{1}{\pi^2} \underbrace{\int_{|y| \geq \delta} \frac{1}{y^2}\, dy}_{\simeq \frac{1}{\delta}} \to 0, \text{ as } T \to \infty.$$

$\leadsto \sup_x |F_T * f(x) - f(x)| \leq \varepsilon + \frac{C}{T} \to 0 \quad \leadsto \quad F_T * f \to f$ uniformly!

Remains to prove: $\displaystyle\int_{-\infty}^{\infty} F_T(x)\,dx = \underbrace{\frac{1}{T}\int_{-\infty}^{\infty}\left(\frac{\sin \pi T x}{\pi x}\right)^2 dx}_{} = 1 \quad \forall T>0$

$$= \{y = Tx,\ dy = T\,dx\} = \int_{-\infty}^{\infty}\left(\frac{\sin \pi y}{\pi y}\right)^2 dy$$

Recall from Module I: $\displaystyle\int_{-1/2}^{1/2} F_N(x)\,dx = \int_{-1/2}^{1/2} \frac{1}{N}\left(\frac{\sin \pi N x}{\sin \pi x}\right)^2 dx = 1 \quad \forall N \geq 1$ (integer)

~~(Fourier series)~~

$$= \underbrace{\int_{-1/2}^{1/2} \frac{1}{N}\left(\frac{\sin \pi N x}{\pi x}\right)^2 dx}_{} + \underbrace{\int_{-1/2}^{1/2} \frac{1}{N}\underbrace{\left(\frac{1}{\sin^2 \pi x} - \frac{1}{(\pi x)^2}\right)}_{} \sin^2 \pi N x \; dx}_{\to 0 \text{ as } N\to\infty}$$

$$= \int_{-N/2}^{N/2} \frac{\sin^2 y\pi}{(y\pi)^2}\,dy \quad\downarrow\quad \int_{-\infty}^{\infty}\frac{\sin^2 \pi y}{(\pi y)^2}\,dy = 1$$

$$\left(\frac{1}{\sin^2 \pi x} - \frac{1}{(\pi x)^2}\right) = \frac{(\pi x - \sin \pi x)(\pi x + \sin \pi x)}{(\pi x \sin \pi x)^2} \rightsquigarrow \text{bnd. on } [-1/2, 1/2]$$

$x=0$ zero of mult 3; $x=0$ zero of mult 1; $x=0$ zero of mult. 4

Lemma: If $\displaystyle\int_{-\infty}^{\infty} |\hat{f}(\xi)|\,d\xi < +\infty$, then $\displaystyle f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\,d\xi \quad \forall x\in\mathbb{R}$

Proof. By our last lemma, $\displaystyle (F_T * f)(x) = \int_{-T}^{T} (1 - |\xi|/T)\,\hat{f}(\xi)\,e^{2\pi i x\xi}\,d\xi \to f(x)$

Spse. $\displaystyle\int_{-\infty}^{\infty} |\hat{f}(\xi)|\,d\xi < +\infty$. Then:

$$\left| \int_{-\infty}^{\infty} \hat{f}(\xi)\,e^{2\pi i x\xi}\,d\xi - \int_{-T}^{T}(1 - |\xi|/T)\,\hat{f}(\xi)\,e^{2\pi i x\xi}\,d\xi \right|$$

$$= \left| \int_{|\xi|>T} \hat{f}(\xi)\,e^{2\pi i x\xi}\,d\xi + \frac{1}{T}\int_{-T}^{T}\hat{f}(\xi)\,|\xi|\,e^{2\pi i x\xi}\,d\xi \right|$$

$\forall \varepsilon > 0$

$$\leq \underbrace{\int_{|\xi|>T} |\hat{f}(\xi)|\,d\xi}_{\to 0,\text{ as } T\to\infty} + \underbrace{\frac{1}{T}\int_{-\varepsilon T}^{\varepsilon T} |\hat{f}(\xi)|\cdot|\xi|\,d\xi}_{\leq\, \varepsilon \int_{-\infty}^{\infty}|\hat{f}(\xi)|\,d\xi} + \underbrace{\int_{|\xi|>\varepsilon T} |\hat{f}(\xi)|\,d\xi}_{\to 0 \text{ as } T\to\infty}$$

$\Rightarrow \displaystyle\overline{\lim_{T\to\infty}} \leq \varepsilon$ (since $\varepsilon > 0$ arbitrary, we are done!)

Plancherel's Theorem

Thm spe. that $f \in \mathcal{C}(\mathbb{R})$ with $\int_{-\infty}^{\infty} |f(x)|\,dx < +\infty$ and $\int_{-\infty}^{\infty} |f(x)|^2 dx < +\infty$.

Then $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$.

Cor. spe $f, g \in \mathcal{C}(\mathbb{R})$ with $\int |f|, \int |f|^2, \int |g|, \int |g|^2 < +\infty$.

Then $\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi$

Proof of Cor.

$$\left\{\begin{array}{l} \int_{-\infty}^{\infty} |f-g|^2 dx = \int_{-\infty}^{\infty} |f|^2 dx + \int_{-\infty}^{\infty} |g|^2 dx - 2\,\mathrm{Re} \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx \\ \qquad\quad \shortparallel \qquad\qquad\qquad \shortparallel \qquad\qquad\qquad \shortparallel \\ \int_{-\infty}^{\infty} |\hat{f}-\hat{g}|^2 d\xi = \int_{-\infty}^{\infty} |\hat{f}|^2 d\xi + \int_{-\infty}^{\infty} |\hat{g}|^2 d\xi - 2\,\mathrm{Re} \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi \end{array}\right.$$

by Thm!

$\rightsquigarrow \mathrm{Re} \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \mathrm{Re} \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi$ (Replace $g \to \lambda g$, $\lambda \in \mathbb{C}$)

$$\Downarrow$$

$\mathrm{Re}\, \bar{\lambda} \cdot \int f(x)\overline{g(x)}\,dx = \mathrm{Re}\, \bar{\lambda} \int \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi \quad \forall \lambda \in \mathbb{C}$

$\rightsquigarrow \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi$. ⨳

Proof of Thm: set $f^*(x) = \overline{f(-x)}$.

Then $\widehat{f^*}(\xi) = \int_{-\infty}^{\infty} \overline{f(-x)}\, e^{-2\pi i x\xi}\,dx = \int_{-\infty}^{\infty} \overline{f(x)\, e^{-2\pi i x \xi}}\,dx = \overline{\hat{f}(\xi)} \quad \forall \xi \in \mathbb{R}$

Obs: If $f_1, f_2 \in \mathcal{C}(\mathbb{R})$ with $\int |f_1|, \int |f_2| < +\infty$, then

$\widehat{(f_1 * f_2)}(\xi) = \hat{f}_1(\xi)\hat{f}_2(\xi) \quad \forall \xi$ (Proof as in the case of Fourier series)

$\rightsquigarrow \widehat{(f * f^*)}(\xi) = |\hat{f}(\xi)|^2 \quad \forall \xi$

By Fejér's Thm: $(f * f^*)(0) = \lim_{T\to\infty} \int_{-T}^{T} (1 - |\xi|/T)\, |\hat{f}(\xi)|^2 d\xi \leq \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$

$$\shortparallel$$

$\int |f(x)|^2 dx$

$\geq \lim_{T} \int_{-\varepsilon T}^{\varepsilon T} (1 - |\xi|/T)\, |\hat{f}(\xi)|^2 d\xi \geq (1-\varepsilon) \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$

$\forall \varepsilon > 0$

Since $\varepsilon > 0$ is arbitrary $\rightsquigarrow \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$.

<u>Lemma</u> Sps. $f \in \mathcal{C}^1(\mathbb{R})$ with $\int_{-\infty}^{\infty} |f(x)|\,dx < +\infty$ and $\int_{-\infty}^{\infty} |f'(x)|\,dx < +\infty$

Then $\widehat{f'}(\xi) = 2\pi i \xi \hat{f}(\xi)$ $\forall \xi$

<u>Proof</u>. $$2\pi i \xi \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) \cdot 2\pi i \xi\, e^{-2\pi i x \xi}\,dx = -\int_{-\infty}^{\infty} f(x) \cdot \frac{d}{dx} e^{-2\pi i x \xi}\,dx$$

$$= \{\text{partial integration}\} = -\Big(\underbrace{\big[f(x) \cdot e^{-2\pi i x \xi}\big]_{-\infty}^{+\infty}}_{=0} - \int_{-\infty}^{\infty} f'(x)\, e^{-2\pi i x \xi}\,dx \Big)$$

$= \widehat{f'}(\xi)$ $\forall \xi \in \mathbb{R}$.

<u>Cor.</u> Sps. $f \in \mathcal{C}^2_{cpt}(\mathbb{R})$. Then $|\hat{f}(\xi)| \le \dfrac{C}{1+\xi^2}$ $\forall \xi$ (for some cnst. $C > 0$)

<u>Proof.</u> Since both f' and f'' have cpt. supports, their integrals are finite.

$\widehat{f''}(\xi) = 2\pi i \xi \widehat{f'}(\xi) = -4\pi^2 \xi^2 \hat{f}(\xi)$ $\forall \xi$ by Lemma.

$\rightsquigarrow$ $\forall \xi \neq 0$ $$|\hat{f}(\xi)| \le \frac{1}{4\pi^2 \xi^2} |\widehat{f''}(\xi)| \le \frac{1}{4\pi^2\xi^2} \int_{-\infty}^{\infty} |f''(x)|\,dx$$

$$|\hat{f}(0)| \le \int_{-\infty}^{\infty} |f(x)|\,dx$$

$\rightsquigarrow$ $|\hat{f}(\xi)| \le \dfrac{C}{1+\xi^2}$, $C = D \cdot \max\left(\int_{-\infty}^{\infty} |f(x)|\,dx, \int_{-\infty}^{\infty} |f''(x)|\,dx\right)$

(for some constant D.

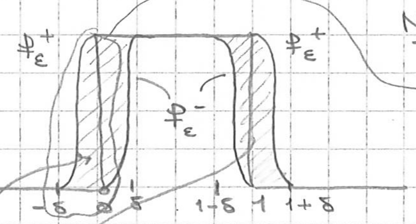
$\boxed{\text{Proof of Proposition on p. 2: Lemma on p. 6 + Lemma on p. 7}}$

<u>Proof of Step 1</u>

(general case identical)

For simplicity, let us only consider the case $I = [0,1]$ $(a = 0,\ b = 1)$.

Fix $\delta > 0$



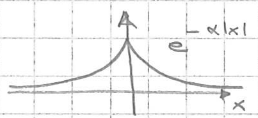
<u>Note:</u>
$$\varphi(x) = \begin{cases} e^{\frac{1}{x+\delta} - \frac{1}{\delta}} & -\delta < x \le 0 \\ 0 & x = -\delta \end{cases}$$
is smooth on $[-\delta, 0]$ including endpts

$$\int_{-\infty}^{\infty} (f_\varepsilon^+ - f_\varepsilon^-)\,dx < 2\delta + 2\delta = 4\delta = \varepsilon$$

Construct f_ε^+ and f_ε^- by pasting together $[0,1]$ and $[\delta, 1-\delta]$ with these types of functions

(here $\underline{\varepsilon = 4\delta}$)

Some transforms



1) • Let $f(x) = e^{-\alpha|x|}$, $\alpha > 0$.

Then: $$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\alpha|x|} \cdot e^{-2\pi i x\xi}\, dx = \int_{-\infty}^{0} \underbrace{e^{\alpha x} \cdot e^{-2\pi i x \xi}}_{e^{(\alpha - 2\pi i \xi)x}}\, dx$$

$$+ \int_{0}^{\infty} \underbrace{e^{-\alpha x} \cdot e^{-2\pi i x \xi}}_{e^{(-\alpha-2\pi i \xi)x}}\, dx = \frac{1}{\alpha - 2\pi i \xi} - \frac{1}{-\alpha - 2\pi i \xi} = \frac{2\alpha}{\alpha^2 + 4\pi^2\xi^2} \quad \forall \xi \in \mathbb{R}$$

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2) Let $f(x) = \frac{1}{\beta^2 + x^2}$, $\beta > 0$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{1}{\beta^2 + x^2} e^{-2\pi i x \xi} = \ldots ?$$

By 1) and Lemma p. 5: $$e^{-\alpha|x|} = \int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + 4\pi^2\xi^2} e^{2\pi i x \xi}\, d\xi \quad \forall x \in \mathbb{R}$$

$$= \{ y = 2\pi\xi,\ dy = 2\pi\, d\xi \} = \frac{\alpha}{\pi} \int_{-\infty}^{\infty} \frac{1}{\alpha^2 + y^2} e^{-2\pi i y \left(-\frac{x}{2\pi}\right)}\, dy \quad \forall x \in \mathbb{R}$$

$$\rightsquigarrow \int_{-\infty}^{\infty} \frac{1}{\alpha^2 + y^2} e^{-2\pi i y x}\, dy = \frac{\pi}{\alpha} e^{-2\pi\alpha|x|} \quad \forall x \in \mathbb{R}$$

$$\rightsquigarrow \hat{f}(\xi) = \frac{\pi}{\beta} e^{-2\pi\beta|\xi|} \quad \forall \xi \in \mathbb{R} \quad (\text{Use } \xi \text{ for } x).$$

$$\boxed{\text{Problem 1 Find } \hat{f} \text{ if } f(x) = \frac{1}{x^2 + 4x + 10}.}$$

Solution: $f(x) = \frac{1}{(x+2)^2 + 6} = g(x+2)$, where $g(x) = \frac{1}{(\sqrt{6})^2 + x^2}$.

$$\rightsquigarrow \hat{f}(\xi) = \int_{-\infty}^{\infty} g(x+2) e^{-2\pi i x \xi}\, dx = \int_{-\infty}^{\infty} g(x) e^{-2\pi i (x-2)\xi}\, dx$$

$$= e^{4\pi i \xi} \hat{g}(\xi) = \frac{e^{4\pi i \xi}\pi}{\sqrt{6}} e^{-2\pi\sqrt{6}|\xi|} \quad \forall \xi \in \mathbb{R}$$

$$\boxed{\text{Problem 2: Compute } I = \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\, dx.}$$

Solution: $$I = \int_{-\infty}^{\infty} \underbrace{\frac{e^{ix}}{1+x^2}}_{f(x)} \underbrace{\frac{1}{1+x^2}}_{\overline{g(x)}}\, dx = \{\text{Plancherel}\} = \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi$$

$\hat{g}(\xi) = \pi e^{-2\pi|\xi|}$, $f(x) = e^{ix} g(x) \rightsquigarrow \hat{f}(\xi) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i x\left(\xi - \frac{1}{2\pi}\right)}\, dx$

$$= \hat{g}\left(\xi - 1/2\pi\right) = \pi e^{-2\pi|\xi - 1/2\pi|}$$