$$\rightsquigarrow I = \int_{-\infty}^{\infty} \pi \cdot e^{-2\pi|\xi|} \cdot \pi e^{-2\pi|\xi - 1/2\pi|} \, d\xi$$

$$= \pi^2 \left(\int_{-\infty}^{0} + \int_{0}^{1/2\pi} + \int_{1/2\pi}^{\infty} \right) = \pi^2 \left(\int_{-\infty}^{0} e^{2\pi(2\xi - 1/2\pi)} \, d\xi \, + \right.$$

$$\left. + \int_{0}^{1/2\pi} e^{-2\pi\xi + 2\pi(\xi - 1/2\pi)} \, d\xi + \int_{1/2\pi}^{\infty} e^{-2\pi(\xi + \xi - 1/2\pi)} \, d\xi \right)$$

$$= \pi^2 \Big(e^{-1} \cdot \underbrace{\int_{-\infty}^{0} e^{4\pi\xi} \, d\xi}_{\frac{1}{4\pi}} + \underbrace{\int_{0}^{1/2\pi} e^{-1} \, d\xi}_{\frac{e^{-1}}{2\pi}} + \underbrace{e^{1} \int_{1/2\pi}^{\infty} e^{-4\pi\xi} \, d\xi}_{\frac{e^{-2}}{4\pi}} \Big)$$

$$= \pi^2 \frac{\left(e^{-1} + 2e^{-1} + e^{-1} \right)}{4\pi} = \pi e^{-1}.$$

3) **Problem 3:** Sps. $\hat{f}(\xi) = \frac{1}{1+|\xi|^3}$. Compute $\int_{-\infty}^{\infty} |f * f'|^2 \, dx$.

Solution Plancherel + Lemma p. 7: $\int_{-\infty}^{\infty} |f * f'|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\xi) \cdot 2\pi \xi i \hat{f}(\xi)|^2 \, d\xi$

$$= 4\pi^2 \int_{-\infty}^{\infty} \frac{\xi^2}{(1+|\xi|^3)^2} \, d\xi = \frac{8\pi^2}{3} \int_{0}^{\infty} \frac{3\xi^2}{(1+\xi^3)^2} \, d\xi = \{ u = \xi^3 \}$$

$$= \frac{8\pi^2}{3} \int_{0}^{\infty} \frac{du}{(1+u)^2} = \frac{8\pi^2}{3} \underbrace{\left[-\frac{1}{(1+u)} \right]_0^{\infty}}_{=1} = \frac{8\pi^2}{3}$$

4) **Problem 4:** Find a function f s.t.

$$e^{-x^4} = \int_{-\infty}^{\infty} f(x-y) e^{-|y|} \, dy \quad \forall x \in \mathbb{R}$$

Solution: Set $g(y) = e^{-|y|}$ and $h(x) = e^{-x^4}$, so that $h = f * g$

and thus $\hat{h}(\xi) = \hat{f}(\xi) \hat{g}(\xi) = \{\text{from above}\} = \frac{2\hat{f}(\xi)}{1+4\pi^2\xi^2} \quad \forall \xi$

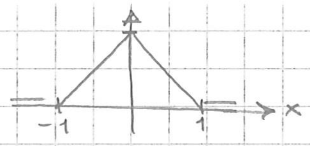
$$\rightsquigarrow \hat{f}(\xi) = (1 + 4\pi^2\xi^2) \tfrac{1}{2} \hat{h}(\xi)$$

$$= \tfrac{1}{2}\hat{h}(\xi) - \tfrac{1}{2}(2\pi i)^2 \hat{h}(\xi) \rightsquigarrow f = \tfrac{1}{2} h - \tfrac{1}{2} h'' \quad \text{(by Lemma earlier)}$$

In our case, $h'(x) = -4x^3 e^{-x^4}$, $h''(x) = -4(3x^2 - 4x^6) e^{-x^4}$

$$\rightsquigarrow f(x) = \tfrac{1}{2}(1 - 12x^2 + 16x^6) e^{-x^4}$$

5) Let $f(x) = \begin{cases} 1-|x| & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$



$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx = \{f \text{ even}\} = 2\int_0^{\infty} f(x) \cos 2\pi x\xi \, dx$$

$$= 2\int_0^1 (1-x)\cos 2\pi x \xi \, dx \underset{\xi \neq 0}{=} 2\left(\left[\frac{\sin 2\pi x \xi}{2\pi\xi}\right]_0^1 - \left(\left[x \cdot \frac{\sin 2\pi x\xi}{2\pi \xi}\right]_0^1 - \right.\right.$$

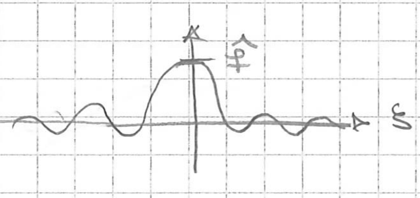
$$\left.\left. - \frac{1}{2\pi\xi}\int_0^1 \sin 2\pi x \xi \, dx \right)\right) = 2\left(\cancel{\frac{\sin 2\pi\xi}{2\pi\xi}} - \left(\cancel{\frac{\sin 2\pi\xi}{2\pi\xi}} - \frac{1}{2\pi\xi}\left[- \frac{\cos 2\pi\xi x}{2\pi\xi}\right]_0^1 \right)\right)$$

$$= 2 \cdot \frac{1}{(2\pi\xi)^2}\left(1 - \cos 2\pi\xi\right) = \left\{ \begin{array}{l} \cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta \\ \rightsquigarrow 1 - \cos 2\theta = 2\sin^2\theta \end{array} \right\}$$

$$= 2 \cdot \frac{1}{(2\pi\xi)^2} \cdot 2\sin^2 \pi\xi = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2 \quad \forall \xi \neq 0.$$

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(x)\, dx = 1.$$

$$\rightsquigarrow \underline{\hat{f}(\xi) = \begin{cases} 1 & \xi = 0 \\ \left(\frac{\sin \pi \xi}{\pi\xi}\right)^2 & \xi \neq 0 \end{cases}}$$



<u>Recall</u> (Additional material Module I)

<u>Poisson's summation formula</u>: If $f \in \mathcal{C}(\mathbb{R})$ with

$$|f(x)| \le \frac{A}{1+x^2} \quad \forall x \quad \text{and} \quad |\hat{f}(\xi)| \le \frac{B}{1+\xi^2} \quad \forall \xi,$$

then:

$$\boxed{\sum_{m\in\mathbb{Z}} f(m) = \sum_{n \in \mathbb{Z}} \hat{f}(n)} \qquad (*)$$

Let $g(x) = \begin{cases} 1-|x| & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$ and set $f_t(x) = t\, g(tx)$ for $t > 0$

Then $\hat{f}_t(\xi) = t\int_{-\infty}^{\infty} g(tx)\, e^{-2\pi i t x \cdot \frac{\xi}{t}}\, dx = \hat{g}(\xi/t) \quad \forall \xi$

If $t \ge 1$, then $(*)$ $\quad t = f_t(0) = \sum_{m \in \mathbb{Z}} f_t(m) = \sum_{n\in\mathbb{Z}} \hat{f}_t(n)$

$$= 1 + \sum_{n \neq 0} \left(\frac{\sin \pi n/t}{\pi n/t}\right)^2 \rightsquigarrow \boxed{t - 1 = 2\sum_{n=1}^{\infty}\left(\frac{\sin \pi n/t}{\pi n/t}\right)^2 \quad \forall t \ge 1.}$$

6) Let $f_a(x) = \begin{cases} \left(\frac{\sin a\pi x}{\pi x}\right)^2 & x \neq 0 \\ a^2 & x = 0 \end{cases}, \quad a > 0.$

$$\hat{f}_a(\xi) = \int_{-\infty}^{\infty} \left(\frac{\sin a\pi x}{\pi x}\right)^2 e^{-2\pi i \xi x}\, dx = \dots ?$$

By 5) + Fejer: $\displaystyle\int_{-\infty}^{\infty} \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2 e^{2\pi i x\xi}\, d\xi = \begin{cases} 1-|x| & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1. \end{cases}$

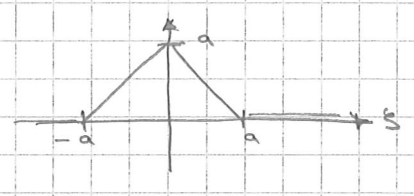
$$= a^2 \int_{-\infty}^{\infty} \left(\frac{\sin a\pi x}{a\pi x}\right)^2 e^{-2\pi i \cdot \frac{\xi}{a}(ax)}\, dx$$

replace $\xi \leftrightarrow x$

$$= a \int_{-\infty}^{\infty} \left(\frac{\sin \pi y}{\pi y}\right)^2 e^{2\pi i \frac{\xi}{a} y}\, dy = a \cdot \begin{cases} 1 - \frac{1}{a}|\xi| & \text{if } |\xi| \le a \\ 0 & \text{if } |\xi| > a \end{cases}$$

(even integrand)

$$= \begin{cases} a - |\xi| & \text{if } |\xi| \le a \\ 0 & \text{if } |\xi| > a \end{cases}$$



Problem 5 Set $\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$ for $t > 0$

Prove that $\theta(t)\sqrt{t} = \theta(1/t)$ $\forall t > 0$ (Jacobi's θ-relation)

Solution Set $f_t(x) = e^{-\pi t x^2}$, by page 8 $\hat{f}_t(\xi) = \frac{1}{\sqrt{t}} e^{-\pi \xi^2 / t}$
$(\alpha = \pi t > 0)$

$\rightsquigarrow$ (Poisson) $\displaystyle \theta(t) = \sum_{n=-\infty}^{\infty} f_t(n) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{t}} e^{-\pi n^2/t} = \frac{1}{\sqrt{t}}\theta(1/t)$ $\forall t > 0$

Problem 6 Show that if $f(x) = e^{-\pi x^2}$, then:

$\forall N \geq 1$ $\forall x_1, \dots, x_N \in \mathbb{R}$ $\forall c_1, \dots, c_N \in \mathbb{C}$

$$\sum_{k,l=1}^{N} c_k \bar{c}_l f(x_k - x_l) \geq 0 \qquad \left(\begin{array}{c} f \text{ is called} \\ \text{positive def.} \end{array}\right)$$

Solution: $\displaystyle f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi$, $\hat{f}(\xi) = e^{-\pi\xi^2}$

$$\rightsquigarrow \underbrace{\sum_{i,j=1}^{N} c_k \bar{c}_l f(x_k - x_l)}_{\geq 0} = \int_{-\infty}^{\infty} \underbrace{\hat{f}(\xi)}_{>0} \underbrace{\left(\sum_{k,l=1}^{N} c_k \bar{c}_l e^{2\pi i (x_k - x_l)\xi}\right)}_{= \left|\sum_{k=1}^{N} c_k e^{2\pi i x_k \xi}\right|^2 \geq 0} d\xi$$

for any $c_1, \dots, c_N \in \mathbb{C}$
$x_1, \dots, x_N \in \mathbb{R}$