Sampling + Oversampling

Def. A unif. continuous + abs. integrable $f : \mathbb{R} \to \mathbb{C}$ is called **band-limited** if $\exists \alpha > 0$ s.t. $\hat{f}(\xi) = 0 \quad \forall |\xi| \geq \alpha/2$ (α = band width)

Ex. $f_\alpha(x) := \left(\frac{\sin(\alpha \pi x/2)}{\pi x} \right)^2 \rightsquigarrow \hat{f}_\alpha(\xi) = \begin{cases} \alpha/2 - |\xi| & \text{if } |\xi| < \alpha/2 \\ 0 & \text{if } |\xi| \geq \alpha/2 \end{cases}$

(see p. 11, Module IV)

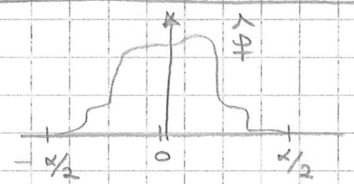
Shannon's sampling theorem

Every band-limited fcn f with band width α can be reconstructed from its values at the points $x_m := \frac{m}{\alpha}$. More precisely,

$$f(x) = \sum_{m=-\infty}^{\infty} f\left(\frac{m}{\alpha}\right) \operatorname{sinc}((\alpha x - m)\pi) \quad \forall x \in \mathbb{R},$$

where:

$$\operatorname{sinc}(y) = \begin{cases} 1 & y = 0 \\ \frac{\sin y}{y} & y \neq 0 \end{cases}$$



Proof: think of $\hat{f}$ as (the restriction of) a α-periodic (unif. cont.) fcn on $\mathbb{R}$.

$$c_{\hat{f}}(m) = \frac{1}{\alpha} \int_{-\alpha/2}^{\alpha/2} \hat{f}(\xi) e^{-\frac{2\pi i \xi m}{\alpha}} d\xi = \frac{1}{\alpha} f\left(-\frac{m}{\alpha}\right) \quad \text{[by Fourier inversion]}$$

* Let us assume that $|f(x)| \leq \frac{C}{1+x^2}$ for some $C > 0$ so that $\sum_{m=-\infty}^{\infty} |c_{\hat{f}}(m)| < +\infty$,

and thus $\hat{f}(\xi) = \frac{1}{\alpha} \sum_{m=-\infty}^{\infty} f\left(-\frac{m}{\alpha}\right) e^{\frac{2\pi i \xi m}{\alpha}} \quad \forall |\xi| \leq \alpha/2$ (see Module I)

Fourier inversion: $f(x) = \int_{\cancel{-\infty}\,-\alpha/2}^{\cancel{\infty}\,\alpha/2} \hat{f}(\xi) e^{2\pi i \xi x} = \frac{1}{\alpha} \sum_{m=-\infty}^{\infty} f\left(-\frac{m}{\alpha}\right) \underbrace{\int_{-\alpha/2}^{\alpha/2} e^{2\pi i \left(\frac{m}{\alpha} + x\right) \xi} d\xi}_{= \frac{\sin \pi(\alpha x + m)}{\pi\left(\frac{m}{\alpha} + x\right)}}$

$$= \sum_{m=-\infty}^{\infty} f\left(\frac{m}{\alpha}\right) \operatorname{sinc} \pi(\alpha x - m) \quad \forall x \in \mathbb{R}.$$

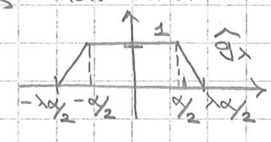
Con: $m \mapsto \operatorname{sinc} \pi(\alpha x - m)$ decays like $\frac{1}{m}$ (very slow)

Q. Is there a way to boost this?

* There are ways to work around this assumption.

Oversampling: Fix $\alpha > 0$ throughout.

Exercise: For $\forall \lambda > 1$, show that $\exists$ unif. cont. + abs. integrable fcn $g_\lambda : \mathbb{R} \to \mathbb{C}$ s.t.



— and g_λ decays like $\frac{1}{x^2}$.

Obs: If f is band-limited with band width α, then $\hat{f} = \hat{f} \cdot \hat{g}_\lambda$ for all $\lambda > 1$.

Idea: Expand $\hat{f}$ in a $\lambda\alpha$-periodic Fourier series:

$$c_{\hat{f}}(m) = \frac{1}{\lambda\alpha} \int_{-\lambda\alpha}^{\lambda\alpha} \hat{f}(\xi) e^{-\frac{2\pi i \xi m}{\lambda\alpha}} d\xi = \frac{1}{\lambda\alpha} f\left(-\frac{m}{\lambda\alpha}\right) \quad \text{(as before)}$$

$$\rightsquigarrow f(x) = \int_{-\cancel{\infty}\,\lambda\alpha/2}^{\cancel{\infty}\,\lambda\alpha/2} \hat{f}(\xi) \hat{g}_\lambda(\xi) e^{2\pi i \xi x} d\xi = \frac{1}{\lambda\alpha} \sum_{m=-\infty}^{\infty} f\left(-\frac{m}{\lambda\alpha}\right) \underbrace{\int_{-\cancel{\lambda\alpha/2}\,\infty}^{\cancel{\lambda\alpha}\,+\infty} \hat{g}_\lambda(\xi) e^{2\pi i \left(x + \frac{m}{\lambda\alpha}\right)\xi} d\xi}_{g_\lambda\left(x + \frac{m}{\lambda\alpha}\right)}$$

$$= \frac{1}{\lambda\alpha} \sum_{m=-\infty}^{\infty} f\left(\frac{m}{\lambda\alpha}\right) g_\lambda\left(x - \frac{m}{\lambda\alpha}\right)$$

fast converging sum!

Sens moral: By oversampling ($\left(\frac{m}{\lambda\alpha}\right)$ are more densely packed than $\left(\frac{n}{\alpha}\right)$) we achieve better convergent sampling formulas.

Playing around with Plancherel

Recall: $\forall f, g : \mathbb{R} \to \mathbb{C}$ unif. cont s.t. $\int |f|, \int |f|^2, \int |g|, \int |g|^2 < +\infty$ we have:

$$\int_{-\infty}^{\infty} f(x) \overline{g(x)}\, dx = \int_{-\infty}^{\infty} \hat{f}(\xi) \overline{\hat{g}(\xi)}\, d\xi.$$

For $n \geq 0$, let $V_n = \{ p(x) e^{-\pi x^2} : p \text{ polynomial of degree} \leq n \}$

Note: $\dim_{\mathbb{C}} V_n = n + 1$.

Claim: If $f(x) = p(x) e^{-\pi x^2}$, $\deg(p) = n$, then $\hat{f}(\xi) = q(\xi) e^{-\pi \xi^2}$ for some polynomial q of degree $= n$.

Proof:
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} p(x) e^{-\pi x^2} \cdot e^{-2\pi i x \xi} dx = \left(\int_{-\infty}^{\infty} p(x) e^{-\pi (x + i\xi)^2} dx \right) e^{-\pi \xi^2}$$

$$= \{ p \text{ is defined on } \mathbb{C}, \text{ not only } \mathbb{R} \} = \left(\int_{\operatorname{Im} z = \xi} p(z - i\xi) e^{-\pi z^2} dz \right) e^{-\pi \xi^2}$$

$$= \left\{ p(z) = \sum_{k=0}^{n} a_k z^k,\ a_n \neq 0 \right\} = \sum_{k=0}^{n} a_k \underbrace{\left(\int_{\operatorname{Im} z = \xi} (z - i\xi)^k e^{-\pi z^2} dz \right)}_{\text{(polynomial in } \xi \text{ of deg} = k)} e^{-\pi \xi^2}$$

$$= q(\xi) e^{-\pi \xi^2}, \text{ where}$$

$$\boxed{q(\xi) = \int_{-\infty}^{\infty} p(x) e^{-\pi (x + i\xi)^2} dx, \quad \xi \in \mathbb{R}.}$$ — polynomial of deg $= n$.

In other words, the Fourier transform $f \mapsto \hat{f}$ induces a linear map $\mathcal{F}_n : V_n \circlearrowleft \ \forall n \geq 0$.

We can identify $V_n \cong \mathrm{Poly}_n$ (Polynomials of deg $\leq n$)

and endow it with the inner product:

$$\langle p_1, p_2 \rangle = \int_{-\infty}^{\infty} p_1(x)\overline{p_2(x)}\, e^{-2\pi x^2}\, dx, \quad \text{for } p_1, p_2 \in \mathrm{Poly}_n.$$

Note that if $p_j e^{-\pi x^2} \xrightarrow{\mathcal{F}} q_j e^{-\pi x^2}$, then

$$\langle q_1, q_2 \rangle = \int_{-\infty}^{\infty} \underbrace{q_1(x) e^{-\pi x^2}}_{= \widehat{p_1 e^{-\pi x^2}}} \cdot \overline{\underbrace{q_2 e^{-\pi x^2}}_{= \widehat{p_2 e^{-\pi x^2}}}}\, dx = \{\text{Plancherel}\}$$

$$= \int_{-\infty}^{\infty} p_1(x) e^{-\pi x^2} \cdot \overline{p_2(x) e^{-\pi x^2}}\, dx = \langle p_1, p_2 \rangle$$

$\rightsquigarrow$ $\mathcal{F}_n : \mathrm{Poly}_n \circlearrowleft$ unitary (preserves inner product)
(abuse of notation)

Linear algebra fact: Every unitary linear map on a finite-dimensional inner product space can be diagonalized

($\iff \exists$ basis of eigenvectors).

Q. How do we diagonalize the Fourier transform $\mathcal{F}_n : \mathrm{Poly}_n \circlearrowleft$?

($\iff$ find the eigenbasis).

Trick: Let $V = \bigcup_{n \geq 0} V_n = \{ p e^{-\pi x^2} : p \text{ polynomial} \} \cong \mathrm{Poly}$ (∞-dim)

$\langle \cdot, \cdot \rangle$ still an inner product on this space (pre-Hilbert — exercise!)

Claim 1: The linear map $Hf = -f'' + 4\pi^2 x^2 \cdot f$, $\forall f \in V$
is self-adjoint on $(V, \langle \cdot, \cdot \rangle)$, i.e. $\langle Hf, g \rangle = \langle f, Hg \rangle \ \forall f, g \in V$
and commutes with $\mathcal{F}$, i.e. $H\mathcal{F}f = \mathcal{F}Hf$, $\forall f \in V$.

Claim 2: $\forall n \geq 0$ $\exists!$ (up to scaling) polynomial p_n of degree n s.t.
s.t. $H\varphi_n = 2\pi(2n+1)\varphi_n$ ($\rightsquigarrow \langle \varphi_k, \varphi_\ell \rangle = 0 \ \forall k \neq \ell$ — by self-adjoint).

where: $\varphi_n = p_n e^{-\pi x^2}$

Hence: $g_n := \mathcal{F}\varphi_n$ $(= q_n \cdot e^{-\pi x^2})$ satisfies $Hg_n = H\mathcal{F}\varphi_n \overset{\text{Claim 1}}{=} \mathcal{F}H\varphi_n \overset{\text{Claim 2}}{=} 2\pi(2n+1)\, g_n$

Since $\deg q_n = n$ $\rightsquigarrow$ $\boxed{\mathcal{F}\varphi_n = \lambda_n \varphi_n \text{ for some } (\lambda_n)}$ to be determined!
and by uniqueness of (φ_n) up to scaling

Proof of Claim 1 :

- Self-adjointness = Tedious partial integration (exercise!)
- Commutes = — " —

Proof of Claim 2 :

Let us write out H's action on polynomial:

Set $f(x) = p(x)e^{-\pi x^2}$

$$f'(x) = (p'(x) - 2\pi x\, p(x))e^{-\pi x^2}$$

$$f''(x) = (p''(x) - 2\pi(p(x) + x p'(x)))e^{-\pi x^2} - 2\pi x(p'(x) - 2\pi x\, p(x))e^{-\pi x^2}$$

$$= (p''(x) - 4\pi x\, p'(x) + (4\pi^2 x^2 - 2\pi)p(x))e^{-\pi x^2}$$

$$\rightsquigarrow Hf(x) = -f''(x) + 4\pi^2 x^2 f(x) = \underbrace{(-p''(x) + 4\pi x\, p'(x) + 2\pi p(x))}_{(Lp)(x)} e^{-\pi x^2}$$

$$\left[\begin{array}{l} H(p_n e^{-\pi x^2}) = \mu_n p_n e^{-\pi x^2} \\ \deg p_n = n \end{array}\right] \iff \boxed{Lp_n = \mu_n p_n}$$

Write $p_n(x) = \sum_{k=0}^{n} a_k x^k$, $a_n \neq 0$

$$Lp_n = \mu_n p_n \iff -a_n n(n-1)x^{n-2} + 4\pi n\, a_n x^n + 2\pi a_n x^n + \text{lower order}$$

$$= \mu_n a_n x^n + \text{lower order}$$

$$\Rightarrow \quad 4\pi n + 2\pi = \boxed{\mu_n = 2\pi(2n+1)}$$

Q. Are there polynomials (p_n), $\deg p_n = n$ s.t.

$$\boxed{-p_n''(x) + 4\pi x\, p_n'(x) + 2\pi p_n(x) = 2\pi(2n+1)p_n(x)} \quad ?$$

Yes! (called Hermite (-de Bruijn) polynomials in the literature)

Exercise : Find p_0, p_1, p_2 (up to scaling)

Problem: How to determine the eigenvalues for $\mathcal{F}\varphi_n = \lambda_n \varphi_n$, λ_n.
where $\varphi_n = p_n e^{-\pi x^2}$?

Trick: We know that $\deg p_n = n$, and

$$\lambda_n p_n(\xi) = \int_{\operatorname{Im} z = \xi} p_n(z - i\xi) e^{-\pi z^2} dz$$

$$\|$$

$$\lambda_n a_n \xi^n + \text{l.o.t.} = a_n \int_{\operatorname{Im} z = \xi} (z - i\xi)^n e^{-\pi z^2} dz + \text{l.o.t.}$$

$$= \{\text{binomial exp.}\} = a_n (-i)^n \xi^n \Big(\underbrace{\int_{\operatorname{Im} z = \xi} e^{-\pi z^2} dz}_{= \int_{-\infty}^{\infty} e^{-\pi (x + i\xi)^2} dx = 1 \ \forall \xi} \Big) + \text{l.o.t.}$$

Since $a_n \neq 0$: $\lambda_n = (-i)^n$ (see Module IV)

$$\Rightarrow \boxed{\mathcal{F}\varphi_n = (-i)^n \varphi_n}$$

Parseval's formula for Hermite fcns

Let us from now on assume that the Hermite-deBruijn polynomials (p_n) have been scaled so that

$$\int_{-\infty}^{\infty} |\varphi_n(x)|^2 dx = \int_{-\infty}^{\infty} |p_n(x)|^2 e^{-2\pi x^2} dx = 1 \quad \forall n \geq 0 \qquad (*)$$

We shall use two properties of (p_n) as black boxes (they are tedious, but not difficult, to prove)

Black box 1: $\exists a > 0$ s.t. $p_n' = a\sqrt{n} \cdot p_{n-1} \quad \forall n \geq 0 \quad (p_{-1} = 0)$

Black box 2: $\exists b > 0$ s.t. $\|\varphi_n\|_\infty = \sup_x |\varphi_n(x)| \leq b \quad \forall n \geq 0$

(Cramer's inequality)

Obs: Black box 1 $\rightsquigarrow$ $\varphi_n'(x) = (p_n'(x) - 2\pi x p_n(x)) e^{-\pi x^2}$

$= a\sqrt{n} \cdot \varphi_{n-1}(x) - 2\pi x \cdot \varphi_n(x) \quad \forall n \geq 0.$

$\rightsquigarrow \forall f \in C(\mathbb{R})$ with $\int_{-\infty}^{\infty} |f|^2 dx$, $\int_{-\infty}^{\infty} |f'|^2 dx < +\infty$:

$$\gamma_n(f) := |\langle f, \varphi_n \rangle| = \frac{1}{a\sqrt{n+1}} |\langle f, \varphi_{n+1}' + 2\pi x \varphi_{n+1} \rangle| \leq \frac{1}{a\sqrt{n+1}} |-\langle f', \varphi_{n+1} \rangle|$$

$$+ \frac{1}{a\sqrt{n+1}} |\langle f, \underbrace{2\pi x \cdot \varphi_{n+1}}_{\mathcal{F}\,\Downarrow\; i\hat{\varphi}_{n+1}' = i(-i)^n \varphi_{n+1}'} \rangle| = \frac{2}{a\sqrt{n+1}} \underbrace{|\langle f', \varphi_{n+1} \rangle|}_{\gamma_{n+1}(f')}$$

$$\rightsquigarrow \gamma_n(f) \leq \frac{4}{a^2 \sqrt{n+1} \cdot \sqrt{n+2}} \gamma_{n+2}(f'') \rightsquigarrow \sqrt{(n+1)(n+2)} \cdot \gamma_n(f) \leq \frac{4}{a^2} \cdot \gamma_{n+2}(f'')$$

$$\rightsquigarrow \sum_{n=1}^{N} \gamma_n(f) = \sum_{n=1}^{N} \sqrt{(n+1)(n+2)} \cdot \frac{1}{\sqrt{(n+1)(n+2)}} \gamma_n(f) \leq$$

(Cauchy-Schwarz)

$$\leq \Big(\underbrace{\sum_{n=1}^{N} \frac{1}{(n+1)(n+2)}}_{\leq C \;\forall n} \Big)^{1/2} \Big(\sum_{n=1}^{N} \frac{16}{a^4} \gamma_{n+2}^2 (f'') \Big)^{1/2}$$

$$\leq \frac{4}{a^2} \Big(\underbrace{\sum_{n=1}^{N} \gamma_{n+2}(f'')^2}_{} \Big)^{1/2}$$

$$\leq \Big(\sum_{n=1}^{\infty} |\langle f'', \varphi_n\rangle|^2 \Big)^{1/2} \leq \int_{-\infty}^{\infty} |f''|^2 dx$$

Bessel's inequality (Module II)

$$\rightsquigarrow \sum_{n=0}^{\infty} |\langle f, \varphi_n\rangle| < +\infty \quad \text{if} \quad \int |f''|^2 dx < +\infty, \quad \int |f'|^2, \int |f^2| < +\infty. \quad (**)$$

Black box 2 + Weierstraß M-test (Module I):

$$\rightsquigarrow \psi_f(x) = \sum_{n=0}^{\infty} \langle f, \varphi_n\rangle \varphi_n(x) \quad \text{converges uniformly } \forall x \in \mathbb{R}.$$

Set $\Delta_f(x) = f(x) - \psi_f(x) = f(x) - \sum_{n=0}^{\infty} \langle f, \varphi_n\rangle \varphi_n(x)$ unif. continuous fcn. $\int |\Delta_f|^2 < +\infty$

$$\rightsquigarrow \langle \Delta_f, \varphi_m\rangle = 0 \quad \forall m \geq 0$$

$$\rightsquigarrow \int \Delta_f(x) x^m e^{-\pi x^2} dx = 0 \quad \forall m \geq 0$$

$$\rightsquigarrow F(t) = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \cdot \int_{-\infty}^{\infty} \Delta_f(x) x^m \cdot e^{-\pi x^2} dx = 0 \quad \forall t \in \mathbb{R}$$

$$= \int_{-\infty}^{\infty} \Delta_f(x) e^{-\pi x^2} \cdot e^{itx} dx$$

$$\Rightarrow \Delta_f(x) e^{-\pi x^2} = 0 \quad \forall x \quad \text{(Fourier inversion)}$$

$$\Rightarrow \Delta_f(x) = 0 \quad \forall x \quad \rightsquigarrow \boxed{f(x) = \sum_{n=0}^{\infty} \langle f, \varphi_n\rangle \varphi_n(x) \quad \forall x}$$

and $\boxed{\int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{n=0}^{\infty} |\langle f, \varphi_n\rangle|^2}$ if $(**)$ holds.

"Parseval's formula"

Heisenberg's Uncertainty Principle

Theorem: suppose (**) holds for f and xf:

$$\left(\int x^2 |f(x)|^2 dx\right)^{1/2} \left(\int \xi^2 |\hat{f}(\xi)|^2 d\xi\right)^{1/2} \geq \frac{1}{\sqrt{\pi}} \left(\int |f(x)|^2 dx\right)^{1/2}$$

(f and $\hat{f}$ cannot both be "concentrated" around 0). Equality $\iff f(x) = c \cdot e^{-\pi x^2}$

$$(**) \rightsquigarrow \quad f(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x) \quad \text{and}: \quad \hat{f}(\xi) = \sum_{n=0}^{\infty} a_n (-i)^n \varphi_n(\xi)$$

$$\left(\sum |a_n| < +\infty \right)$$

$$\boxed{\text{Black box 3}: \quad \sqrt{4\pi} \cdot x \, \varphi_n(x) = \sqrt{n+1} \cdot \varphi_{n+1}(x) + \sqrt{n}\, \varphi_{n-1}(x)}$$

$$\rightsquigarrow \quad x f(x) = \sum_{n=0}^{\infty} \left(\frac{\sqrt{n+1}\, a_n + \sqrt{n} \cdot a_{n-1}}{\sqrt{4\pi}} \right) \varphi_n(x)$$

$$\xi \cdot \hat{f}(\xi) = \sum_{n=0}^{\infty} (-i)^{n+1} \left[\sqrt{n+1}\, a_n + \sqrt{n} \cdot a_{n-1} \right] \varphi_n(\xi)$$

Parseval's formula:

$$\int x^2 |f(x)|^2 dx = \frac{1}{4\pi} \sum_{n=0}^{\infty} \underbrace{\left| \sqrt{n+1} \cdot a_n + \sqrt{n} \cdot a_{n-1} \right|^2}_{= (n+1)|a_n|^2 + n|a_{n-1}|^2 + 2\sqrt{n(n+1)} \cdot \mathrm{Re}\, a_n \overline{a_{n-1}}}$$

$$\int \xi^2 |\hat{f}(\xi)| d\xi = \frac{1}{4\pi} \sum_{n=0}^{\infty} \underbrace{\left| \sqrt{n+1}\, a_n - \sqrt{n}\, a_{n-1} \right|^2}_{= (n+1)|a_n|^2 + n|a_{n-1}|^2 - 2\sqrt{n(n+1)}\, \mathrm{Re}\, a_n \overline{a_{n+1}}}$$

$$\rightsquigarrow \quad \int x^2 |f(x)|^2 dx + \int \xi^2 |\hat{f}(\xi)|^2 d\xi$$

$$= \frac{1}{2\pi} \sum_{n=0}^{\infty} \left((n+1)|a_n|^2 + n|a_{n-1}|^2 \right) = \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1)|a_n|^2$$

$$\geq \frac{1}{\pi} \sum_{n=0}^{\infty} |a_n|^2 = \frac{1}{\pi} \int |f(x)|^2 dx$$

$= 0 \iff a_n = 0 \quad \forall n \geq 1 \iff f(x) = a_0 \varphi_0(x) = c \cdot e^{-\pi x^2}$

for some c.

$$\rightsquigarrow \quad \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx + \int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2 d\xi \geq \frac{1}{\pi} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Trick: Replace f with $f_\lambda(x) = \sqrt{\lambda}\, f(\lambda x)$, $\lambda > 0$

Note: i) f_λ still satisfies $(**)$. $\quad y = \lambda x \quad dy = \lambda\, dx$

ii) $$\int_{-\infty}^{\infty} |f_\lambda(x)|^2\,dx = \int_{-\infty}^{\infty} |f(x)|^2\,dx \quad \forall \lambda$$

$$\int_{-\infty}^{\infty} x^2 |f_\lambda(x)|^2\,dx = \left\{ y = \lambda x,\ dx = \frac{dy}{\lambda} \right\} = \frac{1}{\cancel{\lambda}}\frac{1}{\lambda^2} \int \cancel{\lambda}\, y^2 |f(y)|^2\,dy$$

$$\int \xi^2 |\hat{f}_\lambda(\xi)|^2\,d\xi = \left\{ \hat{f}_\lambda(\xi) = (\hat{f})_{1/\lambda}(\xi) \right\} = \lambda^2 \int \xi^2 |\hat{f}(\xi)|^2\,d\xi$$

$$\rightsquigarrow \quad \underbrace{\frac{1}{\lambda^2}\int_{-\infty}^{\infty} x^2 |f(x)|^2\,dx + \lambda^2 \int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2\,d\xi}_{\text{Minimize in } \lambda\ !} \geq \frac{1}{\pi}\int_{-\infty}^{\infty} |f(x)|^2\,dx \quad \forall \lambda > 0$$

$$\rightsquigarrow \quad \int_{-\infty}^{\infty} x^2 |f(x)|^2\,dx \cdot \int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2\,d\xi \geq \left(\frac{1}{\pi}\int_{-\infty}^{\infty} |f(x)|^2\,dx \right)^2$$

equality $\Leftrightarrow$ $f(x) = c\,e^{-\pi x^2}$