Laplace transform

$f: \mathbb{R} \to \mathbb{C}$ Riemann-integrable.

Def. • f right-sided if $f(t)=0 \ \forall t<0$

• f exponentially integrable if $\exists \alpha > 0$ s.t. $f_\alpha(t) := e^{-\alpha t} f(t)$ is absolutely integrable.

If f is right-sided + exp. integrable : $\forall \rho > \alpha$

$$\hat{f}_\rho(\xi) = \int_{\cancel{-\infty}\,0}^{\infty} f(t) e^{-\rho t} \cdot e^{-2\pi i \xi t}\, dt = \int_0^\infty f(t) e^{-\underbrace{(\rho + 2\pi i \xi)}_{=s} t}\, dt$$

$$=: (\mathcal{L}f)(s) \quad , \begin{pmatrix} \text{exists if} \\ Re(s) > \alpha \end{pmatrix}$$

If f is continuous, then (by Fejér) $\mathcal{L}f$ determines f uniquely

—"— (piecewise) continuous, then (by Fejér) $\mathcal{L}f$ —"—
(except at points of discontinuities)

Also: If $\hat{f}_\rho$ is abs. integrable for some ρ, then

$$f(t) = e^{\rho t} \cdot \int_{-\infty}^{\infty} \hat{f}_\rho(\xi) e^{2\pi i \xi t}\, d\xi = \int_{Re(s)=\rho} \mathcal{L}f(s) e^{st} \frac{ds}{i} \quad \forall t \in \mathbb{R}$$

f, g right-sided + exp. int.

Main properties: Set $F = \mathcal{L}f$ and $G = \mathcal{L}g$ (where they both exist).

1) $f(t) = g'(t) \rightsquigarrow F(s) = s\,G(s) - g(0)$ [Partial integration]

2) $f(t) = g''(t) \rightsquigarrow F(s) = s^2 G(s) - s\,g(0) - g'(0)$

3) $f(t) = g(t-t_0) \rightsquigarrow F(s) = e^{-st_0} G(s)$ [Variable subst.]
$t_0 \geq 0$

4) $f * g \rightsquigarrow F \cdot G$ [Change of order of integr.]

5) $f(t) = \begin{cases} \frac{t^n}{n!} e^{\lambda t} & , t \geq 0 \\ 0 & , t < 0 \end{cases} \rightsquigarrow F(s) = \dfrac{1}{(s-\lambda)^{n+1}}$ ($n \geq 0$ integer.)

Applications to LTI-system: $y = h * x$, h right-sided + exp. int

(2)

Def. Stable $\iff \int_0^\infty |h(t)|\,dt < +\infty$.

Thm (Stability criterium) Sps. $H(s) = \frac{P(s)}{Q(s)} e^{-\mu s}$, $\mu \geq 0$.

P, Q polynomials
grad P ≤ grad Q
P, Q no common roots

Stable $\iff$ All roots of Q have negative real parts

"Proof": $H(s) = H_0(s) e^{-\mu s} \iff h(t) = h_0(t-\mu)$ by 3)

$$\int_0^\infty |h(t)|\,dt < +\infty \iff \int_0^\infty |h_0(t)|\,dt < +\infty$$

$$H_0(s) = \sum_{k=1}^{r} \left(\sum_{j=1}^{m_k} \frac{c_{kj}}{(s-s_k)^j} \right), \quad s_1, \ldots, s_r = \text{roots of } Q$$

(Partial fraction)

m_k = multiplicity of s_k

$\left(\rightsquigarrow \sum_{k=1}^{r} m_k = \text{degree of } Q \right)$

By 5: $h_0(t) = \sum_{k=1}^{r} \left(\sum_{j=0}^{m_k-1} \frac{c_{k,j+1}}{j!} t^j e^{-s_k t} \right)$

abs. integrable $\iff$ $\mathrm{Re}(s_k) < 0 \quad \forall k = 1, \ldots, r$. (easy!)