Solution Exam MMG710/TMA362

Jan 9 / 2019

1a) See Cor 2 on p. 3 in "Main results" on the homepage.

b) Let $f(x) = e^{-|x|}$ and note that

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-|x|} \cdot e^{-2\pi i \xi x}\, dx = \int_{-\infty}^{0} e^{x(1-2\pi i \xi)}\, dx$$

$$+ \int_{0}^{\infty} e^{-x(1+2\pi i \xi)}\, dx = \left[\frac{e^{x(1-2\pi i \xi)}}{(1-2\pi i \xi)} \right]_{-\infty}^{0} - \left[\frac{e^{-x(1+2\pi i \xi)}}{1+2\pi i \xi} \right]_{0}^{\infty}$$

$$= \frac{1}{1-2\pi i \xi} + \frac{1}{1+2\pi i \xi} = \frac{2}{1+4\pi^2 \xi^2}$$

Since both f and $\hat{f}$ are absolutely integrable,

$$\text{a)} \Rightarrow f(x) = e^{-|x|} = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-2\pi i \xi x}\, d\xi \overset{\text{(even)}}{=} 4 \cdot \int_{0}^{\infty} \frac{1}{1+4\pi^2 \xi^2} \cos 2\pi \xi x \, d\xi$$

$$= \{ u = 2\pi \xi,\ du = 2\pi\, d\xi \} = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{1+u^2} \cos u \cdot x \, du \quad \forall x \in \mathbb{R}.$$

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2) Set $g(x) = e^{-\pi x^2}$ so that $\hat{g}(\xi) = e^{-\pi \xi^2}$.

Note: $g'(x) = -2\pi x \cdot e^{-\pi x^2} \overset{\text{"Formulas"}}{\rightsquigarrow} \hat{g'}(\xi) = 2\pi i \xi\, \hat{g}(\xi) = 2\pi i \xi\, e^{-\pi \xi^2}$

Note: $f(x) = (ax+b)\, e^{-\pi x^2}$

$$= -\frac{a}{2\pi} g'(x) + b \cdot g(x) \rightsquigarrow \hat{f}(\xi) = (\underbrace{-ia}_{=c}\xi + \underbrace{b}_{=d})\, e^{-\pi \xi^2}$$

$$\rightsquigarrow \begin{pmatrix} c \\ d \end{pmatrix} = \underbrace{\begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}}_{=T} \begin{pmatrix} a \\ b \end{pmatrix} \rightsquigarrow \boxed{\text{Eigenvals} = \{-i, 1\}}$$

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3) $$F(s) = \int_{0}^{\infty} f(x) e^{-sx}\, dx = \int_{1}^{\infty} (1-x)\, e^{-(s+1)x}\, dx$$

$$= \{ u = x-1,\ du = dx,\ u: 0 \to \infty \} = \left(-\int_0^{\infty} u\, e^{-(s+1)u}\, du \right) e^{-(s+1)}$$

$$= \boxed{-\frac{1}{(s+1)^2} e^{-(s+1)}} \quad \text{by "Hint"}.$$

(for $\mathrm{Re}(s) > -1$.)

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4a) <u>Odd extension</u>: $f(x) = x(1-2x) \quad 0 \leq x < 1/2$

and $f(-x) = -f(x)$ for $-1/2 < x \leq 0$.

$$c_f(m) = \int_{-1/2}^{1/2} f(x)\, e^{-2\pi i m x}\, dx = -i \int_0^{1/2} x(1-2x) \sin 2\pi m x\, dx$$

$$= -i \left(\frac{(-1)^{m+1}}{4\pi m} - 2 \cdot \left(\frac{(-1)^{m+1}}{8\pi m} - \frac{(1-(-1)^m)}{4\pi m^3} \right) \right)$$

$$= \begin{cases} 0 & m = 2n \\ \frac{-i}{\pi(2n-1)^3} & m = 2n-1 \end{cases}, \quad n \geq 1. \quad (c_f(-m) = -c_f(m))$$

$$f(x) = \sum_{m=-\infty}^{\infty} c_f(m)\, e^{2\pi i m x} = \sum_{m=1}^{\infty} c_f(m) \left(e^{2\pi i m x} - e^{-2\pi i m x} \right)$$

$$= 2i \sum_{m=1}^{\infty} c_f(m) \sin 2\pi m x = \boxed{\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin 2\pi(2n-1)x}$$

b) With the usual "<u>Ansatz</u>":

$$u(x,t) = \sum_{m=1}^{\infty} b_m e^{-4\pi^2 m^2 t} \sin 2\pi m x,$$

where $\sum_{m=1}^{\infty} b_m \sin 2\pi m x = x(1-2x) \quad \forall\, 0 \leq x \leq 1/2$.

By a) $b_m = 0$ if $m = 2n$ and $b_m = \frac{2}{\pi} \cdot \frac{1}{(2n-1)^3}$ if $m = 2n-1$

so: $$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{2}{\pi} \cdot \frac{1}{(2n-1)^3} e^{-4\pi(2n-1)^3 t} \sin 2\pi(2n-1)x}$$