Weierstraß example

Thm Suppose $f(x) = \sum_{k=1}^{\infty} a_k \cos(2^k \pi x)$, $\sum_{k=1}^{\infty} |a_k| < +\infty$.

If $\exists x_0$ and c s.t. $|f(x_0 - y) - f(y)| \leq c|y|$ $\forall y$, then

there exists $c' > 0$ s.t. $|a_k| \leq c' \cdot k \cdot 2^{1-k}$ $\forall k \geq 1$.

Cor. If $1 < \beta < 2$, then $f(x) = \sum_{k=1}^{\infty} \beta^{-k} \cos(2^k \pi x)$ is continuous,

but not differentiable at any point. "Weierstraß classical example"

Proof. There is no constant s.t. $\beta^{-k} \leq c' \cdot k \cdot 2^{-k}$ $\forall k \geq 1$, if $1 < \beta < 2$.

Recall: Fejér kernel $F_N(x) = \sum_{|m| \leq N} \left(1 - \frac{|m|}{N}\right) e^{2\pi i m x} = \frac{1}{N} \left(\frac{\sin \pi N x}{\sin \pi x}\right)^2$

Obs: $F_N(x) \leq \begin{cases} N & |x| \leq 1/N \\ \frac{1}{4Nx^2} & 1/N \leq x \leq 1/2 \end{cases}$ $(*)$

Let $f(x) = \sum_{k=1}^{\infty} a_k \cos(2^k \pi x)$, $\sum_{k=1}^{\infty} |a_k| < +\infty$ and note

that $c_f(m) = \begin{cases} 0 & m \neq \pm 2^k \\ \frac{a_k}{2} & m = \pm 2^k \end{cases}$.

In particular: If we define $I_k := \int_0^1 (f(x_0 - y) - f(x_0)) F_{2^{k-1}}(y) e^{2\pi i \cdot 2^k y} dy$,

then $I_k = \int_0^1 f(x_0 - y) F_{2^{k-1}}(y) e^{2\pi i 2^k y} dy - 0 = \sum_m c_f(m) \int_0^1 e^{2\pi i m (x_0 - y)} F_{2^{k-1}}(y) e^{2\pi i 2^k y} dy$

$= e^{2\pi i 2^k x_0} c_f(2^k) = \frac{1}{2} e^{2\pi i \cdot 2^k x_0} a_k \rightsquigarrow |I_k| = \frac{1}{2}|a_k|$, $\forall k$. $(**)$

(Special form of f.)

On the other hand: For $k \geq 2$,

1-periodicity!

$$|I_k| \le \int_0^1 |f(x_0-y)-f(x_0)|\, F_{2^{k-1}}(y)\,dy = \int_{-1/2}^{1/2} |f(x_0-y)-f(x_0)|\, F_{2^{k-1}}(y)\,dy$$

$$\le \int_{-1/4}^{+1/4} \; + \; 2\|f\|_\infty \int_{1/4}^{1/2} F_{2^{k-1}}(y)\,dy$$

assume $|f(x_0-y)-f(x_0)| \le C|y| \ \forall y$

$$\le 2C \int_0^{1/2^{k-1}} y\, F_{2^{k-1}}(y)\,dy + 2C \int_{1/2^{k-1}}^{1/4} y \cdot F_{2^{k-1}}(y)\,dy$$

$$+ \; 2\|f\|_\infty \cdot \int_{1/4}^{1/2} F_{2^{k-1}}(y)\,dy$$

(D₁ ind. of k!)

(*)

$$\le 2C \cdot 2^{k-1} \left(\frac{1}{2^{k-1}}\right)^2 \cdot \frac{1}{2} + \underbrace{\frac{2C}{4 \cdot 2^{k-1}} \int_{1/2^{k-1}}^{1/4} \frac{1}{y}\,dy}_{\le \frac{D_1 k}{2^k}}$$

$$+ \; \frac{2\|f\|_\infty}{4 \cdot 2^{k-1}} \underbrace{\int_{1/4}^{1/2} \frac{dy}{y}}_{D_2}$$

$$\le \frac{c'' k}{2^k} \quad \forall k \qquad \left(c'' = 4C + D_1 + D_2\|f\|_\infty\right)$$

$$\rightsquigarrow \quad |a_k| \le \frac{2 \cdot c' \cdot k}{2^k} =: \frac{c' k}{2^k}$$