Fourier analysis fall 2008. Exercises 11.

1. Folland, Exercise 3.5.1.

2. Folland, Exercise 3.5.3.

3. Folland, Exercise 3.5.7. Also expand the function $-x$ as a sum of the eigenfunctions. What can you say about convergence?

4. Consider the eigenvalue problem $f'' + \lambda f = 0$, with boundary values $f(0) = 2f'(0)$, $f(4) = 2f'(4)$. Find all eigenfunctions and eigenvalues. Also expand the constant function 1 as a sum of the eigenfunctions. What can you say about convergence?

Answers:

3. $-x = \sum_n \frac{4\cos(\nu_n)}{\nu_n(1+\cos^2\nu_n)} \sin(\nu_n x)$, with ν_n as in Folland. Convergence is uniform on $0 \le x \le 1$.

4. The positive eigenvalues are $\lambda_k = k^2\pi^2/16$, $k = 1, 2, 3, \ldots$, with eigenfunctions (not normalized)

$$u_k(x) = \frac{k\pi}{2}\cos\left(\frac{k\pi x}{4}\right) + \sin\left(\frac{k\pi x}{4}\right).$$

There is also a negative eigenvalue $\lambda_0 = -1/4$, with eigenfunction $u_0(x) = e^{x/2}$. The eigenvalue expansion is

$$1 = \frac{2}{e^2+1}e^{x/2} + \frac{16}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)(4+\pi^2(2k+1)^2)} \times \left(\frac{(2k+1)\pi}{2}\cos\left(\frac{(2k+1)\pi x}{4}\right) + \sin\left(\frac{(2k+1)\pi x}{4}\right)\right).$$

Theorem 3.10 guarantees convergence in $L^2([0,4])$. If one moves the first term from the right-hand to the left-hand side what remains is a Fourier series, and one can use the theory of Fourier series to prove that convergence is pointwise in the open interval $0 < x < 4$.