Fourier analysis fall 2008. Exercises 2.

1. Let $f(t) = t$ and $g(t) = \cos(t)$. Consider them as causal function, that is, redefine them so they vanish for $t < 0$. Compute the convolution $f * g$ and check that $\mathcal{L}(f)\mathcal{L}(g) = \mathcal{L}(f * g)$.

2. Let $f(t) = e^{-t^2}$. Compute $f * f$. You will need that

$$\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}.$$

3. Let

$$g(t) = \begin{cases} t, & 0 < t < 1, \\ 2 - t, & 1 < t < 2, \\ 0, & t > 2. \end{cases}$$

   (a) Express g using Heaviside's function.

   (b) Compute the Laplace transform $\mathcal{L}(g)$.

   (c) Solve the initial value problem

$$x'' + 2x' + x = g(t), \qquad x(0) = x'(0) = 0.$$

4. Solve the equation

$$x'' - 3x' + 2x = e^t, \qquad x(0) = x'(0) = 0$$

   in two ways. First, by taking Laplace transform; second, by appling the formula involving convolution with the fundamental solution (impulse response).

5. Let $\{t\}$ denote the fractional part of t, e.g. $\{\pi\} = 0.141592\dots$. Compute the Laplace transform of $\{t\}$. (This type of saw-tooth wave is used in ancient technology such as synthesizers and TVs — look up saw-tooth on Wikipedia.)

Answers (may contain typos)

1. The convolution is $(f * g)(t) = 1 - \cos t$.

2. $(f * f)(t) = \sqrt{\pi/2}\, e^{-t^2/2}$.

3. (b) $G(s) = \dfrac{1}{s^2}(1 - 2e^{-s} + e^{-2s})$; (c) $x(t) = f(t) - 2H(t-1)f(t-1) + H(t-2)f(t-2)$, where $f(t) = (t+2)e^t + t - 2$.

4. $x(t) = e^{2t} - (t+1)e^t$.

5. $\dfrac{1 + s - e^s}{s^2(1 - e^s)}$.