Fourier analysis fall 2008. Exercises 4.

1. Let f be the 2π-periodic function defined by $f(x) = e^{\cos(x^2)}$ for $0 \le x < 2\pi$. What is the value of its Fourier series at $x = 4\pi$? This does not require any computation!

2. Exercise 2.2.5 in Folland.

3. Exercises 2.4.3, 2.4.7, 2.4.11 in Folland. (Use Table 1.)

4. Let f be the 2π-periodic function defined by $f(x) = \cosh(x) = (e^x + e^{-x})/2$ for $|x| \le \pi$. Express it as a Fourier series. Compute

$$\sum_{k=0}^{\infty} \frac{1}{k^2+1}.$$

5. Let f be the 2π-periodic function defined by $f(x) = \cos(ax)$ for $|x| \le \pi$, where a is not an integer. Express it as a Fourier series. Deduce the useful identity

$$\pi \cot(\pi a) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{a+n}, \qquad a \notin \mathbb{Z}.$$

6. Gibbs phenomenon: Exercise 2.6.1 in Folland.

Answers:

1. $\dfrac{e^{\cos(4\pi^2)} + e}{2}.$

4. The Fourier series is

$$\frac{\sinh(\pi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{inx}}{1+n^2}$$

and the sum is

$$\frac{\pi}{2}\coth(\pi) - \frac{1}{2}.$$

5. The Fourier series is

$$\frac{a\sin(\pi a)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{inx}}{a^2-n^2}.$$