Fourier analysis fall 2008. Exercises 5.

1. Solve the heat conduction problem

$$\begin{cases} u'_t = 3u''_{xx}, & 0 \le x \le \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, & t \ge 0, \\ u(x,0) = \sin x \cos(4x), & 0 \le x \le \pi. \end{cases}$$

2. Solve the heat conduction problem

$$\begin{cases} u'_t = u''_{xx}, & 0 \le x \le \frac{\pi}{2}, \quad t > 0, \\ u(0,t) = u'_x(\pi/2,t) = 0, & t \ge 0, \\ u(x,0) = x(\pi - x), & 0 \le x \le \frac{\pi}{2} \end{cases}$$

   (Use Exercise 1.3.7 and Table 2.1 in Folland,).

3. A large steak with thickness l and heat diffusivity a is to be cooked in an oven with temperature T. It is taken directly from the fridge, where the temperature is 0. The temperature inside the steak can then be described by

$$\begin{cases} u'_t = au''_{xx}, & 0 \le x \le l, \quad t > 0, \\ u(0,t) = u(l,t) = T, & t > 0, \\ u(x,0) = 0, & 0 \le x \le l. \end{cases}$$

   (a) Determine, as a series, $u(x,t)$. To apply Fourier's method, one should first shift the temperature scale so that T corresponds to 0 (equivalently, introduce the function $v(x,t) = u(x,t) - T$).

   (b) Suppose all terms in the series except the first can be ignored. When will the steak be cooked if that happens when the temperature is everywhere at least $T/4$? How much longer does it take if the thickness of the steak is doubled?

4. Find a solution to the wave equation $u''_{tt} = 4u''_{xx}$, valid for all x and t, such that $u(x,0) = x^2$, $u'_t(x,0) = x$. Hint: Look for it in the form $f(x+2t) + g(x-2t)$.

5. Folland, Exercise 2.5.5. Also consider the following question: Where should the string be plucked if we do not wish to hear the sixth overtone (corresponding to the seventh term in the Fourier series)? Motivation: The first five overtones are close to tones in the usual scale. The sixth overtone is somewhere between two half-tones, and thus sounds false (at least to people accustomed to European music).

6. Folland, Exercies 2.5.6. In this example, can we choose δ so the sixth overtone is avoided?

Answers:

1. $\frac{1}{2}(e^{-75t}\sin(5x) - e^{-27t}\sin(3x))$.

2. $\frac{8}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}e^{-(2k+1)^2 t}\sin((2k+1)x)$.

3. (a) $T - \frac{4T}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}e^{-(2k+1)^2\pi^2 at/l^2}\sin((2k+1)\pi x/l)$.

   (b) The cooking time is approximately $\frac{l^2}{a\pi^2}\ln\left(\frac{16}{3\pi}\right)$, so a twice as thick steak takes four times as long to cook.

4. $x^2 + 4t^2 + xt$.

5. Second question: In one of the points $a = kl/7$, $k = 1, 2, \ldots, 6$. Note that these are the nodes of the sixth overtone.

6. Second question: Yes, take $\delta = kl/7$, $k = 1, 2, \ldots, 6$.