Fourier analysis fall 2008. Exercises 6.

1. In the Lecture we looked at the function

$$g_N(x) = 2\sum_{n=1}^{N} \frac{\sin(nx)}{n} - (\pi - x),$$

which measured the distance between the saw-tooth function and its Nth Fourier approximation. We saw that $kx_N = k\pi/(N+1/2)$ is the kth smallest positive critical point of g_N. In the Lecture we only considered the case $k = 1$; here we want to look at general k. Let

$$M_k = \lim_{N\to\infty} g_N(kx_N).$$

(a) Show that $M_k = 2\int_0^{k\pi} \frac{\sin x}{x}\, dx - \pi$.

(b) Show that $\int_{n\pi}^{(n+2)\pi} \frac{\sin x}{x}\, dx$ is positive for n even and negative for n odd.

Hint: Divide the integral as $\int_{n\pi}^{(n+1)\pi} + \int_{(n+1)\pi}^{(n+2)\pi}$, and then change x to $x + \pi$ in the second integral.

(c) Prove that

$$M_1 > M_3 > M_5 > \cdots > 0,$$
$$M_2 < M_4 < M_6 < \cdots < 0,$$

and that $M_k \to 0$ as $k \to \infty$. Use (a), (b) and the fact that $\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.

2. Find a 2π-periodic solution to the equation

$$x'' + 2x' + x = \cos(t).$$

Try to do this in several different ways! Also find the solution to

$$x'' + 2x' + x = \cos(t), \qquad x(0) = x'(0) = 0$$

and observe that it approaches the periodic solution as t grows.

Answer to Exercise 2: $x(t) = \frac{1}{2}\sin t$, $x(t) = \frac{1}{2}\sin t - \frac{1}{2}te^{-t}$.