Fourier analysis fall 2008. Exercises 7.

1. Show that, in an inner product space, $\|0\| = 0$.

2. Show that, in an inner product space, if $u_n \to u$ and $v_n \to v$ (in norm), then $\langle u_n, v_n \rangle \to \langle u, v \rangle$. (Turn page for hint.)

3. Show that, if $(e_k)_{k=1}^{\infty}$ is a complete orthonormal system in an inner product space, then

$$\langle f, g \rangle = \sum_{k=1}^{\infty} \langle f, e_k \rangle \langle e_k, g \rangle.$$

   (Turn page for hint.)

4. Show that $p(x) = \frac{1}{\sqrt{2}}$ and $q(x) = \sqrt{\frac{3}{2}}x$ form an orthonormal system, with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx.$$

5. Use the previous exercise to determine the constants a, b that minimize the integral

$$\int_{-1}^{1} |e^x - ax - b|^2\, dx.$$

6. Find a second degree polynomial r such that, in the notation of Exercise 4, p, q, r form an orthonormal system.

7. Apply Parseval's formula to the 2π-periodic function $f(x) = x^2$, $|x| < \pi$. Use the result to compute $\sum_1^\infty \frac{1}{n^4}$.

8. Define $J_n(x)$ through the Fourier series

$$e^{ix \sin(t)} = \sum_{n=-\infty}^{\infty} J_n(x) e^{int}$$

   (these are called Bessel functions). Compute, for $x \in \mathbb{R}$,

$$\sum_{n=-\infty}^{\infty} |J_n(x)|^2.$$

Hints:

2. Write $\langle u, v\rangle - \langle u_n, v_n\rangle = \langle u, v - v_n\rangle + \langle u - u_n, v_n\rangle$.

3. Use Exercise 2.

Answers:

5. $a = 3e^{-1}$, $b = (e - e^{-1})/2$.

6. $r(x) = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$

7. $\frac{\pi^4}{90}$.

8. 1, for any x.