Fourier analysis fall 2008. Exercises 9.

1. Using that $1/(x^2+1)$ has Fourier transform $\pi e^{-|\xi|}$, find the Fourier transforms of

   (a) $\dfrac{1}{x^2+6x+13}$,

   (b) $\dfrac{x}{(x^2+1)^2}$.

2. Use Fourier transform to find a function f such that

$$\int_{-\infty}^{\infty} f(t-y)e^{-|y|}\,dy = 2e^{-|t|} - e^{-2|t|}.$$

3. For a and b positive, use Fourier transform to compute the integrals

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(at)\sin(b(u-t))}{t(u-t)}\,dt$,

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(at)\sin(bt)}{t^2}\,dt$.

4. Using that $e^{-|x|}\cos x$ has Fourier transform $2(\xi^2+2)/(\xi^4+4)$ (see Exercise 8.1(e)), compute

$$\int_{-\infty}^{\infty} \frac{(x^2+2)^2}{(x^4+4)^2}\,dx.$$

5. Find (as an integral) a solution to the heat equation $u_{xx} = u_t$ for $t > 0$, $x \in \mathbb{R}$, where $u(x,t) = 1$ for $|x| < 1$ and 0 else.

6. Show that, if f is even, then under suitable conditions

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F(\xi)\cos(\xi x)\,d\xi,$$

   where

$$F(\xi) = \int_0^{\infty} f(x)\cos(\xi x)\,dx.$$

   Also give a corresponding formula for odd functions.

Answers:

1. (a) $\frac{1}{2}\pi e^{3i\xi-2|\xi|}$, (b) $-\frac{1}{2}\pi i\xi e^{-|\xi|}$.

2. $f(x) = \frac{3}{2}e^{-2|x|}$.

3. (a) $\pi\frac{\sin(cu)}{u}$, (b) πc, where $c = \min(a, b)$.

4. $3\pi/8$.

5. $u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-1}^{1} e^{-(x-y)^2/4t}\, dy$.

6. See Folland, page 238.