Chalmers tekniska högskola och Göteborgs universitet
Matematik
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Further notes on Laplace transform

Convolution

The *convolution* (Swedish: faltning) of two functions is defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\,ds.$$

We can also write this as a line integral

$$\int_{x+y=t} f(x)g(y)\,d\sigma,$$

which makes it obvious that

$$f * g = g * f.$$

One can verify that

$$(f * g) * h = f * (g * h),$$

for instance by showing that both sides equal the surface integral

$$\iint_{x+y+z=t} f(x)g(y)h(z)\,d\Sigma.$$

See also Folland, Section 7.1.

In the context of Laplace transform, we usually work with functions defined for $t \geq 0$. Given two such functions f, g, extend them to the real line by defining $f(t) = g(t) = 0$ for $t < 0$. Then their convolution is given by

$$(f * g)(t) = \int_0^t f(s)g(t-s)\,ds.$$

In particular, convolution is well-defined on the class $\mathcal{C}$, consisting of piecewise continuos functions of exponential order on $[0, \infty[$.

Proposition: Let f and g be in $\mathcal{C}$, with $\mathcal{L}f = F$, $\mathcal{L}g = G$. Then

$$\mathcal{L}(f * g) = F(s)G(s).$$

Proof: We write

$$F(s)G(s) = \int_0^{\infty} e^{-xs} f(x)\,dx \int_0^{\infty} e^{-ys} g(y)\,dy = \iint_{x,y\geq 0} e^{-(x+y)s} f(x)g(y)\,dxdy.$$

We now integrate over line segments $x + y = t$, $x, y \geq 0$ (draw a picture!). If you are comfortable with this, just write the integral as

$$\int_0^\infty e^{-ts} \left(\int_{x+y=t} f(x)g(y)\, d\sigma \right) dt = \mathcal{L}(f * g)(t).$$

Otherwise, the change of variables $x + y = t$, $y = u$ (which has Jacobian 1) gives

$$\int_0^\infty e^{-ts} \left(\int_0^t f(t-u)g(u)\, du \right) dt = \mathcal{L}(f * g)(t).$$

A few words on systems and signals

Laplace transform is widely used by engineers working with signal processing or system analysis. Even if you are not an engineer, it is useful to have an idea of this viewpoint.

Basically, in engineering a "system" means something that transforms an "input signal" u to an "output signal" x. For instance, think of an electronic circuit with resistors, capacitors etc. The input can then be the voltage in a variable power source, and the output the current measured in some other part of the circuit. Many systems can be modelled by linear ordinary differential equations

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_0 x = u.$$

(One can also have derivatives of u on the right-hand side. Moreover, one often has several coupled equations or, equivalently, x and u can be vector-valued. However, we stick to the situation above for simplicity.) If a_i and b_i are constants, the system is called time-invariant.

It is very common to consider input signals such as triangular waves or square waves, which are not given by a simple closed "formula". For such right-hand sides, the elementary Ansatz method for solving ODEs is tedious, since the line must be divided into many sub-intervals which are studied separately. With Laplace transform, computations can be kept to a minimum.

Piecewise defined functions

In order to compute with functions such as square waves, it is useful to express them in terms of Heaviside's function

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}$$

The value of $H(0)$ is usually chosen as 1, 1/2, 0 or is left un-defined. Since we eventually want to put H into integrals, the choice seldom matters.

Please draw the graphs for a function $f(t)$, the function $f(t)H(t-a)$ and the function $f(t-a)H(t-a)$. Note that $f(t)H(t-a)$ can be thought of as "activating" f at time $t = a$.

To get the function $f(t-a)H(t-a)$ we take the graph for $t > 0$ and translate it to the interval $t > a$. Keep these pictures in mind.

For $a \geq 0$, there is a simple rule for the Laplace transform of $f(t-a)H(t-a)$. Indeed,

$$\int_0^\infty f(t-a)H(t-a)e^{-st}\,ds = \int_a^\infty f(t-a)e^{-st}\,ds = \int_0^\infty f(t)e^{-s(t+a)}\,ds = e^{-as}F(s).$$

This is the "second shift rule"

$$\boxed{\begin{array}{ll} f(t) & f(t-a)H(t-a)\ (a \geq 0) \\ F(s) & \qquad e^{-as}F(s) \end{array}}\,.$$

Example: Calculate the Laplace transform of the function

$$f(t) = \begin{cases} 0, & t < 1, \\ (t-1)^2, & 1 \leq t < 2, \\ 1, & t \geq 2. \end{cases}$$

Of course, it is not hard to compute

$$\int_0^\infty f(t)e^{-st}\,dt = \int_1^2 (t-1)^2 e^{-st}\,dt + \int_2^\infty e^{-st}\,dt$$

directly, but we illustrate a somewhat more systematic way. We first express f as

$$f(t) = (t-1)^2 H(t-1) + \left(1 - (t-1)^2\right) H(t-2).$$

For instance, the second term can be read as "at time $t = 2$ activate the function 1 and turn off the function $(t-1)^2$". We then write

$$1 - (t-1)^2 = 2t - t^2 = -(t-2)^2 - 2(t-2),$$

which gives

$$f(t) = (t-1)^2 H(t-1) - (t-2)^2 H(t-2) - 2(t-2)H(t-2)$$

and, by the second shift rule,

$$F(s) = \frac{2}{s^3}e^{-s} - \frac{2}{s^3}e^{-2s} - \frac{2}{s^2}e^{-2s}.$$

Note that, even though you might dismiss f as an "un-natural" function, F is certainly very natural!

Example: Solve the differential equation

$$x'' + 3x' + 2x = f(t), \qquad x(0) = x'(0) = 0, \qquad t \geq 0,$$

where $f(t) = 1$ for $0 \leq t < 1$ and 0 else.

Since $t \geq 0$ we can write

$$f(t) = 1 - H(t-1).$$

Thus, we obtain

$$\begin{aligned}
& (s^2 + 3s + 2)X(s) = \frac{1 - e^{-s}}{s} \\
\iff \quad & X(s) = \frac{1 - e^{-s}}{s(s+1)(s+2)} \\
\iff \quad & X(s) = (1 - e^{-s}) \left(\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)} \right) \\
\iff \quad & x(t) = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} - H(t-1) \left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)} \right) \\
& \quad = \begin{cases} \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}, & 0 \leq t < 1, \\ (e-1)e^{-t} + \frac{1}{2}(1 - e^2)e^{-2t}, & t \geq 1. \end{cases}
\end{aligned}$$

Impulse response

Suppose we replace the square pulse f in the preceeding example with the pulse

$$f_\varepsilon(t) = \begin{cases} \dfrac{1}{\varepsilon}, & 0 \leq t < \varepsilon, \\ 0, & t > \varepsilon \end{cases}$$

(draw a picture!). What happens when $\varepsilon \to 0$?

Note that the area under the graph is constant equal to 1. If we think of f_ε as a force, then this area is *impulse*, that is the change of momentum caused by the force. The limiting case $\varepsilon \to 0$ corresponds to the impulse 1 applied momentarily at time $t = 0$. One can think of the limit as a "function" δ which satisfies

$$\delta(t) = \begin{cases} \infty, & t = 0, \\ 0, & t \neq 0, \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

Unfortunately, no such function exists, but nobody can forbid us to think about it! It is usually called the impulse function (by engineers) or the Dirac delta function (by mathematicians). One can give meaning to δ within the framework of *distributions*, but we will not go into that. We take the more pedestrian view-point that any statement about δ is an abbreviation for a statement about the functions f_ε as $\varepsilon \to 0$.

Let F_ε denote the Laplace transform of f_ε. We have

$$f_\varepsilon(t) = \frac{1}{\varepsilon} - \frac{1}{\varepsilon} H(t-\varepsilon),$$

$$F_\varepsilon(s) = \frac{1}{\varepsilon s} - \frac{e^{-\varepsilon s}}{\varepsilon s} = \frac{1-e^{-\varepsilon s}}{\varepsilon s} \to 1 \qquad (\varepsilon \to 0).$$

This suggests that one should put

$$\mathcal{L}(\delta) = 1. \tag{1}$$

Computing formally, we now consider the system

$$\begin{cases} a_n x^{(n)} + \cdots + a_0 x = \delta, \\ x(0) = x'(0) = \cdots = x^{(n-1)}(0) = 0. \end{cases}$$

That is, we start from a system at rest and then apply an impulse at time $t = 0$. Taking Laplace transform, using (1), gives

$$X(s) = \frac{1}{a_n s^n + \cdots + a_0}.$$

Expanding $X(s)$ as a partial fraction, one can in principle compute the inverse Laplace transform $x(t)$. It is called the *impulse response* (by engineeers) or the *fundamental solution* (by mathematicians).

Alternatively, the impulse response can be introduced without invoking the delta "function" as follows.

Proposition: The impulse response, as defined above, can also be defined as the solution to

$$\begin{cases} a_n x^{(n)} + \cdots + a_0 x = 0, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad x^{(n-1)}(0) = \dfrac{1}{a_n}. \end{cases}$$

Proof: Taking Laplace transform, the initial condition $x^{(n-1)}(0) = 1/a_n$ gives rise to a single non-zero term and we get

$$a_n\left(s^n X(s) - \frac{1}{a_n}\right) + a_{n-1}s^{n-1}X(s) + \cdots + a_0 X(s) = 0 \qquad \Longrightarrow \qquad X(s) = \frac{1}{a_n s^n + \cdots + a_0}.$$

The general inhomogeneous problem

Now look at the problem

$$\begin{cases} a_n x^{(n)} + \cdots + a_0 x = u, \\ x(0) = x'(0) = \cdots = x^{(n-1)}(0) = 0, \end{cases} \tag{2}$$

where u is continuous and of exponential order. Moreover, let h denote the impulse response and $H(s) = 1/(a_n s^n + \cdots + a_0)$ its Laplace transform. We then have

$$(a_n s^n + \cdots + a_0)X(s) = U(s)$$

or

$$X(s) = H(s)U(s).$$

Since multiplication of Laplace transforms corresponds to convolution, this gives the integral formula

$$x(t) = (h * u)(t) = \int_0^t h(s)u(t-s)\,ds \tag{3}$$

for the solution to (2). This result is of great theoretical importance. For instance, it is the starting point for the important method of Green's functions, see Folland, Chapter 10. We remark that (3) is in fact valid for any continuous u (even when the Laplace transform of u does not make sense), as one can verify directly.