Chalmers tekniska högskola och Göteborgs universitet
Matematik
Hjalmar Rosengren
September 25, 2008

# $L^2$ theory of Fourier series

The main facts in Folland, Chapter 3, are Theorem 3.4 and Theorem 3.5. These are formulated in terms of $L^2$ spaces, which require Lebesgue integrals. We prefer to give the corresponding statements for Riemann integrals, but at the end we briefly discuss the more complete $L^2$ theory.

## Complex inner product spaces

In the proof of Bessel's inequality, we used the notation

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx. \tag{1}$$

You may have noticed that we only needed some formal properties of this pairing, not the fact that it is given by an integral or even that $f$ and $g$ are functions. In fact, as we will explain, Bessel's inequality can be generalized to general inner product spaces.

I assume you know what a complex vector space is. (It is a set $V$ with two operations $V\times V \to V$ and $\mathbb{C}\times V \to V$, usually written $(u,v)\mapsto u+v$ and $(\alpha,u)\mapsto \alpha u$, satisfying some familiar-looking axioms.) As long as one considers finite-dimensional vector spaces, there is really only the example $\mathbb{C}^n$. An infinite-dimensional example is the space of polynomials in one complex variable. Another one is the space of complex-valued functions on $\mathbb{R}$, or some subspace such as the $2\pi$-periodic piecewise continuous functions.

An inner product on a complex vector space is a map $V\times V\to\mathbb{C}$, written $(u,v)\mapsto\langle u,v\rangle$, such that (for all $u,v,w\in V$ and $\alpha,\beta\in\mathbb{C}$)

$$\langle \alpha u+\beta v, w\rangle = \alpha\langle u,w\rangle + \beta\langle v,w\rangle,$$

$$\langle u, \alpha v+\beta w\rangle = \bar\alpha\langle u,v\rangle + \bar\beta\langle u,w\rangle,$$

$$\langle u,v\rangle = \overline{\langle v,u\rangle},$$

$$\langle u,u\rangle \geq 0.$$

Once we have an inner product, we can introduce the norm

$$\|u\| = \sqrt{\langle u,u\rangle}.$$

We do *not* assume that the inner product is positive definite, so one can have $\|u\|=0$ for some vectors $u\neq 0$. Usually, one then speaks of "seminorm" rather than norm.

An example of an inner product space is $\mathbb{C}^n$ with the pairing

$$\langle x,y\rangle = x_1\bar y_1+\cdots+x_n\bar y_n.$$

Another one is (1), on some suitable space of functions.

In any inner product space, one has the triangle equality

$$\|u + v\| \le \|u\| + \|v\|,$$

Cauchy-Schwarz' inequality

$$|\langle u, v\rangle| \le \|u\| \cdot \|v\|$$

and Pythagoras' theorem: If $\langle u_j, u_k\rangle = 0$ for $j \neq k$, then

$$\|u_1 + \cdots + u_n\|^2 = \|u_1\|^2 + \cdots + \|u_n\|^2.$$

In fact, the proofs usually given for $\mathbb{C}^n$ (or $\mathbb{R}^n$) work in general.

Finally, having a norm we have a notion of convergence. We say that $x_n \to x$ in norm if

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

For (1), convergence in norm means that

$$\lim_{n\to\infty} \int_0^{2\pi} |f_n(x) - f(x)|^2 \, dx = 0.$$

One can show that uniform convergence implies convergence in norm, but pointwise convergence neither implies nor is implied by convergence in norm, see Folland page 72.

## Orthonormal systems

Let us reformulate our proof of Bessel's inequality in more abstract language. Thus, let $V$ be an inner product space (not necessarily a Hilbert space). Suppose that $e_1, \dots, e_n \in V$ is an orthonormal set, that is,

$$\langle e_j, e_k\rangle = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

Let $f \in V$ be arbitrary. Then,

$$u = \sum_{k=1}^{n} \langle f, e_k\rangle e_k$$

is the orthogonal projection of $f$ on the space $W \subseteq V$ spanned by $e_1, \dots, e_n$. Indeed,

$$\langle u, e_m\rangle = \sum_{k=1}^{n} \langle f, e_k\rangle\langle e_k, e_m\rangle = \langle f, e_m\rangle, \qquad 1 \le m \le n,$$

so $f - u$ is orthogonal to all the vectors $e_1, \dots, e_n$, and thus to the whole space $W$. Now let $v = \sum_{k=1}^{n} \alpha_k e_k$ be an arbitrary vector in $W$. Then $f - u$ is orthogonal to $u - v$, so Pythagoras' theorem gives

$$\|f - v\|^2 = \|f - u\|^2 + \|u - v\|^2. \tag{2}$$

Note that, again by Pythagoras' theorem,

$$\|u - v\|^2 = \sum_{k=1}^{n} |\langle f, e_k\rangle - \alpha_k|^2. \tag{3}$$

When $v = 0$, (2) and (3) give

$$\|f\|^2 = \|f - u\|^2 + \sum_{k=1}^{n} |\langle f, e_k\rangle|^2, \tag{4}$$

which shows that

$$\sum_{k=1}^{n} |\langle f, e_k\rangle|^2 \le \|f\|^2.$$

If we can extend the finite orthonormal system $(e_k)_{k=1}^n$ to an infinite orthonormal system $(e_k)_{k=1}^\infty$ (which requires that $V$ has infinite dimension), then we can let $n \to \infty$ and conclude Bessel's inequality

$$\sum_{k=1}^{\infty} |\langle f, e_k\rangle|^2 \le \|f\|^2.$$

(In the version we gave previously, the system was written $(e_k)_{k=-\infty}^\infty$, but that is a simple matter of relabelling.)

From (4) we also obtain the following criterion for equality in Bessel's inequality.

**Proposition 1:** *For $(e_k)_{k=1}^\infty$ an orthonormal system in an inner product space $V$ and for $f \in V$, the following two conditions are equivalent:*

(i) $\displaystyle\sum_{k=1}^{\infty} |\langle f, e_k\rangle|^2 = \|f\|^2$,

(ii) $\displaystyle\lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} \langle f, e_k\rangle e_k \right\| = 0.$

Condition (ii) can also be expressed as

$$f = \sum_{k=1}^{\infty} \langle f, e_k\rangle e_k, \qquad \text{with convergence in norm.}$$

**Definition:** *An orthonormal system $(e_k)_{k=1}^\infty$ is called* complete *if the two equivalent conditions in* Proposition 1 *hold for all $f \in V$.*

Sometimes, a complete orthonormal system is simply called an orthogonal basis, but note that it is not a "basis" in the usual sense of linear algebra (when vectors can be expressed as *finite* linear combination of basis elements).

So far we only considered the case $v = 0$ of (2), but it is also interesting to vary $v$. As a function of $v$, (3) clearly assumes its minimum if $\alpha_k = \langle f, e_k\rangle$. Thus, the same is true for the left-hand side of (2). This leads to the following (Folland, Corollary 3.1).

**Proposition 2:** *For $(e_k)_{k=1}^n$ an orthonormal system in an inner product space and $f$ an arbitrary vector, the distance*

$$\left\| f - \sum_{k=1}^{n} \alpha_k e_k \right\|$$

*is minimized if and only if $\alpha_k = \langle f, e_k \rangle$.*

In the context of Fourier series, this means that the quantity

$$\int_0^{2\pi} \left| f(x) - \sum_{n=-N}^{N} c_n e^{inx} \right|^2 dx$$

is minimized precisely when $c_n$ are Fourier coefficients of $f$.

## Completeness of Fourier series

The following result is extremely important.

**Theorem:** *The system $(e^{inx})_{n=-\infty}^{\infty}$ is a complete orthonormal system in the space of Riemann integrable functions on $[0, 2\pi]$, with scalar product (1). In particular, for any such function $f$ with Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, we have*

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 \qquad \text{(Parseval's formula)}$$

*and*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \text{with convergence in norm,}$$

*that is,*

$$\lim_{N\to\infty} \int_0^{2\pi} \left| f(x) - \sum_{n=-N}^{N} c_n e^{inx} \right|^2 dx = 0.$$

We need the following lemma.

**Lemma:** *For any Riemann integrable function $f$ on $[0, 2\pi]$ and any $\varepsilon > 0$, one can find a piecewise $C^1$ and continuous function $g$ such that $\|f - g\| < \varepsilon$.*

**Rough sketch of proof:** Riemann integrable means that functions can be approximated well by piecewise constant functions (step functions). Step functions can be approximated well by piecewise linear functions (simply replace each jump by a very steep line segment). Here, "well" means with respect to $L^1$ norm

$$\int_0^{2\pi} |f(x) - g(x)|\, dx$$

rather than $L^2$ norm (with $|f(x) - g(x)|^2$). But for bounded functions on bounded intervals, $L^1$ estimates imply $L^2$ estimates. The proof can be completed along these lines, and one finds that one can even choose $g$ piecewise linear.

**Proof of Theorem:** Let $\varepsilon > 0$ and choose $g$ as in the Lemma with

$$\|f - g\| < \varepsilon.$$

Introduce the notation $S_N(f) = \sum_{n=-N}^{N} c_n e^{inx}$, with $c_n$ Fourier coefficients of $f$. Since the Fourier series for $g$ converges uniformly (Folland, Thm. 2.5) it converges in norm, so if $N$ is big enough

$$\|g - S_N(g)\| < \varepsilon.$$

Proposition 2 above gives

$$\|f - S_N(f)\| \leq \|f - S_N(g)\|.$$

The triangle inequality now gives

$$\|f - S_N(f)\| \leq \|f - g\| + \|g - S_N(g)\| < 2\varepsilon.$$

Since $\varepsilon$ is arbitrary, we conclude that

$$\lim_{N \to \infty} \|f - S_N(f)\| = 0,$$

which is criterion (ii) for completeness of an orthogonal system.

To summarize, we have the following convergence criteria for Fourier series. The Fourier series of a $2\pi$-periodic function $f$ converges to $f$

- pointwise, if $f$ is piecewice $C^1$, and we define $f$ at the jump points as the average of the left and right limit;
- uniformly, if $f$ is piecewice $C^1$ and continuous;
- in norm, if $f$ is Riemann integrable on $[0, 2\pi]$.

## Hilbert spaces

The results discussed above can be generalized to the setting of *Hilbert spaces*, but to understand that properly one needs Lebesgue integration, which is an improvement of Riemann integration ususally discussed in more advanced courses.

A Hilbert space is an inner product space satisfying two additional axioms. First, the norm should be positive definite, which means that

$$\|u\| = 0 \quad \Longrightarrow \quad u = 0.$$

Second, the norm should be complete. This means that if $(u_n)_{n=1}^{\infty}$ is a sequence in $V$ such that

$$\lim_{m,n \to \infty} \|u_n - u_m\| = 0$$

(this is called a Cauchy sequence), then there exists a vector $u \in V$ such that

$$\lim_{n \to \infty} \|u_n - u\| = 0.$$

In many ways, Hilbert spaces are similar to $\mathbb{C}^n$, but they need not be finite-dimensional.

Without going into details, we mention that given an inner product space, there is a natural way to construct a Hilbert space from it. First, if the norm is not positive definite, one identifies any two vectors $u$ and $v$ with $\|u - v\| = 0$. (If you are familiar with quotient spaces, this is an example.) After this identification, the inner product is positive definite.

For instance, the space of piecewise continuous functions on $[0, 2\pi]$ with inner product (1) is not positive definite. Just consider the function $f(x)$ that is 1 at $x = \pi$ and 0 otherwise. It has norm 0 but is not identically zero. The quotient construction means that we should identify two piecewise continuous functions if they are equal except at a finite number of points. In particular, $f$ is identified with the zero function. Many times before we have said things like "the value of the function at the jump point does not matter". This means that we should really work in the quotient space.

After we have made the norm positive definite, there is a way of making it complete by enlarging the space, very roughly speaking by adding all missing limits of Cauchy sequences as new elements. This enlarged space is a Hilbert space. (The method is exactly the same as one of the ways to construct $\mathbb{R}$ from $\mathbb{Q}$.)

If we start from the space of continuous functions on $[0, 2\pi]$ with inner product (1), then the corresponding Hilbert space is denoted $L^2([0, 2\pi])$. It is a remarkable fact that the elements of $L^2([0, 2\pi])$ can be viewed as functions, with inner product given by the same formula (1). The space $L^2([0, 2\pi])$ includes all Riemann integrable functions, but also some which are not, so to make sense of the integral (1) it must be interpreted as a so called Lebesgue integral. We should also note that (1) is not positive definite on Lebesgue integrable functions, so some functions in $L^2([0, 2\pi])$ are identified. (**Example:** The function which is 1 at all rational $x$ and 0 at all irrational $x$ is not Riemann integrable. However, it is Lebesgue integrable on any interval with integral 0. As an element of $L^2([0, 2\pi])$ it is identified with the zero function.)

The theory of Fourier series on $L^2([0, 2\pi])$ is very elegant. The system $(e^{inx})_{n=-\infty}^{\infty}$ is still complete, so for each $f \in L^2([0, 2\pi])$, the Fourier coefficients $(c_n)_{n=-\infty}^{\infty}$ satisfies

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Let $\ell^2(\mathbb{Z})$ denote the space of sequences $(c_n)_{n=-\infty}^{\infty}$ such that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty.$$

This is a Hilbert space, with scalar product

$$\left\langle (c_n)_{n=-\infty}^{\infty}, (d_n)_{n=-\infty}^{\infty} \right\rangle = \sum_{n=-\infty}^{\infty} c_n \bar{d}_n.$$

It turns out that the correspondence between $f \in L^2([0, 2\pi])$ and its Fourier coefficients $(c_n)_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$ is a bijection, and since it preserves the norm the two Hilbert spaces can be identified.