

# Fourier analysis (MMG710/TMA362)

**Time:** 2008-10-25, 08.30–13.30

**Tools:** No calculator or handbook is allowed

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**Grades:** Each problem gives 4 points. For MMG710 grades are G (12-17 points) and VG (18-24 points). For TMA362 grades are 3 (12-14 points), 4 (15-17 points) and 5 (18-24 points).

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- 1 (a) Show that the Fourier transform of  $f(x) = e^{-|x|}$  is  $\hat{f}(\xi) = 2/(\xi^2 + 1)$ .  
(b) Use the result of (a) to compute, for  $a \in \mathbb{R}$ ,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2 + 1} dx.$$

- 2 Explain Fourier's method (separation of variables and superposition) using the example

$$\begin{aligned} u_t' &= u_{xx}'' & t > 0, \quad 0 < x < \pi, \\ u_x'(0, t) &= u_x'(\pi, t) = 0, & u(x, 0) &= f(x). \end{aligned}$$

Also find explicitly the solution in the case  $f(x) = 1 + \cos(3x)$ .

- 3 Find numbers  $c_n$  such that

$$\sum_{n=1}^{\infty} c_n \sin(nx) = \begin{cases} x, & 0 < x < \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < x < \pi. \end{cases}$$

What is the sum of the series for  $x = \pi/2$ ?

- 4 Let

$$f(t) = \begin{cases} \sin t, & 0 \leq t \leq \pi, \\ 0, & \text{else.} \end{cases}$$

- (a) Compute the Laplace transform of  $f$ .  
(b) Solve the initial value problem

$$x'(t) + x(t) = f(t), \quad x(0) = 0.$$

- 5 Formulate and prove a theorem on uniform convergence of Fourier series.  
6 Derive Poisson's integral formula for the upper half-plane  $\{(x, y) \in \mathbb{R}^2; y > 0\}$ . You may use that  $f(x) = 1/(x^2 + a^2)$  has Fourier transform  $\hat{f}(\xi) = \pi e^{-a|\xi|}/a$ . Also find a bounded harmonic function  $u(x, y)$  in the upper half-plane with boundary values

$$u(x, 0) = \begin{cases} 1, & |x| < 1, \\ 0, & \text{else.} \end{cases}$$

**Good luck!**

Hjalmar