Fourier analysis fall 2009. Exercises Thursday 15 October.

1. Let $f \in L^2(\mathbb{R})$ be given by $f(t) = \sin(at)\sin(bt)/t$, with a and b positive. Show that f is band-limited in the sense that $\hat{f}(\omega) = 0$ for $|\omega| > a + b$. You may use that standard properties of the Fourier transform remain valid for L^2 functions, even though we have not gone into this is detail.

2. If δ and Ω are two numbers with $0 < \pi/\Omega < \delta$, find a function $f \in L^2(\mathbb{R})$, such that $\hat{f}(\omega) = 0$ for $|\omega| \geq \Omega$ and $f(n\delta) = 0$ for $n \in \mathbb{Z}$, but $f \neq 0$ as an element of $L^2(\mathbb{R})$. This shows that the Shannon–Nyquist sampling distance $\Delta t = \pi/\omega_{\text{max}}$ is the largest possible. **Hint:** Use the previous Exercise.

3. Exercise 7.3.8 in Folland.

4. Give explicitly the ODE corresponding to the third order Butterworth filter.

5. Give explicitly the zeroes $s_1, \ldots, s_n$ of the characteristic polynomial of the nth degree Butterworth filter. What happens to $\max(\text{Re}(s_k))$ as n grows? What does this mean for the transient solution?

6. Determine the finite (discrete) Fourier transform of the sequence $(1, 0, 0, -1)$. Check the inversion formula.

7. Give a version of Plancherel's formula for finite Fourier transform.

Answers:

2. $f(t) = \sin(\pi t/\delta)\sin((\Omega - \pi/\delta)t)/t.$

4. $x^{(3)} + 2cx'' + 2c^2x' + c^3x = c^3u.$

5. $s_k = ce^{\frac{\pi i}{2}(1+\frac{2k-1}{n})}$; $\max(\mathrm{Re}(s_k)) = c\cos\left(\frac{\pi}{2}\left(1+\frac{1}{n}\right)\right) \to 0$ as $n \to \infty$.

6. $(0, 1-i, 2, 1+i)$.

7. $\sum_{k=1}^{n} |a_k|^2 = \frac{1}{N}\sum_{k=1}^{n} |\hat{a}_k|^2.$