Fourier analysis. Additional exercises.

1. (a) Consider the wave equation $u''_{tt} = c^2 u''_{xx}$. Show that a solution defined in all of $\mathbb{R}^2$ can be written $u(x,t) = f(x+ct) + g(x-ct)$, that is, as the sum of two waves travelling along the x-axis with speed c in opposite directions.

   (b) Show that the standing wave $u(x,t) = \sin(x)\sin(ct)$ is a solution of the wave equation. What does it *look* like? Decompose it as a sum of two waves as in (a).

   (c) Find a solution to the wave equation $u''_{tt} = 4u''_{xx}$, valid for all x and t, such that $u(x,0) = x^2$, $u'_t(x,0) = x$.

2. Consider the vibrating string with fixed endpoints on $0 \le x \le \pi$. Supppose that the initial conditions are $u(x,0) = \sin(2x)$, $u'_t(x,0) = 3\sin(5x)$. What is the solution $u(x,t)$?

3. If you know the Fourier coefficients of f, what can you say about the Fourier coefficients of $f(x-a)$ and of $e^{ikx}f(x)$ (where a is real and k is integer)? Compare with the shift rules for Laplace transform.

4. When f and g are 2π-periodic Riemann integrable functions, define their convolution by

$$(f * g)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(y)g(x-y)\,dy.$$

   Denoting Fourier coefficients by $c_n(f)$, show that $c_n(f * g) = c_n(f)c_n(g)$.

5. Let f be the 2π-periodic function defined by $f(x) = e^{\cos(x^2)}$ for $0 \le x < 2\pi$. What is the value of its Fourier series at $x = 4\pi$?

6. Determine the complex Fourier series of the 2π-periodic function that equals $x(x^2 - \pi^2)$ in $[-\pi, \pi]$. What is the sum of the series at the points 2π and $3\pi/2$?

7. Let f be the 2π-periodic function defined by $f(x) = \cosh(x) = (e^x + e^{-x})/2$ for $|x| \le \pi$. Express it as a Fourier series. Compute

$$\sum_{k=0}^{\infty} \frac{1}{k^2+1}.$$

8. Let f be the 2π-periodic function defined by $f(x) = \cos(ax)$ for $|x| \le \pi$, where a is not an integer. Express it as a Fourier series. Deduce the identity

$$\pi\cot(\pi a) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{a+n}, \qquad a \notin \mathbb{Z}.$$

9. Show that
$$\frac{\sin x}{x} = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos(nx), \qquad 0 < x < \pi,$$
where
$$b_n = \frac{1}{\pi} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin x}{x}\, dx.$$
Use this result to compute
$$\int_0^{\infty} \frac{\sin x}{x}\, dx.$$

10. Expand the function $\cos x$ as a sine series on the interval $(0, \frac{\pi}{2})$. Use the result to compute
$$\sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2}.$$

11. Solve the heat conduction problem
$$\begin{cases} u'_t = 3u''_{xx}, & 0 \le x \le \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, & t \ge 0, \\ u(x,0) = \sin x \cos(4x), & 0 \le x \le \pi. \end{cases}$$

12. Solve the heat conduction problem
$$\begin{cases} u'_t = u''_{xx}, & 0 \le x \le \frac{\pi}{2}, \quad t > 0, \\ u(0,t) = u'_x(\pi/2,t) = 0, & t \ge 0, \\ u(x,0) = x(\pi - x), & 0 \le x \le \frac{\pi}{2} \end{cases}$$

13. A large steak with thickness l and heat diffusivity a is to be cooked in an oven with temperature T. It is taken directly from the fridge, where the temperature is 0. The temperature inside the steak can then be described by
$$\begin{cases} u'_t = au''_{xx}, & 0 \le x \le l, \quad t > 0, \\ u(0,t) = u(l,t) = T, & t > 0, \\ u(x,0) = 0, & 0 \le x \le l. \end{cases}$$

(a) Determine, as a series, $u(x,t)$. Since the boundary conditions are inhomogeneous, one should shift the temperature scale so that T corresponds to 0 (equivalently, introduce the function $v(x,t) = u(x,t) - T$).

(b) Suppose all terms in the series except the first can be ignored. When will the steak be cooked if that happens when the temperature is everywhere at least $T/4$? How much longer does it take if the thickness of the steak is doubled?

14. Addendum to Folland, Exercise 2.5.5: Where should the string be plucked if we do not wish to hear the sixth overtone (corresponding to the seventh term in the Fourier series)? Motivation: The first five overtones are close to tones in the usual scale. The sixth overtone is somewhere between two half-tones, and thus sounds false (at least to people accustomed to European music).

15. Addendum to Folland, Exercise 2.5.6: Can we choose δ so the sixth overtone is avoided?

16. Let $f(t) = 1 - t^2$ for $|t| \leq 1$ and let f be 2-periodic. Determine a bounded solution to

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x > 0, \quad -\infty < t < \infty, \\ u(0,t) = f(t), & -\infty < t < \infty. \end{cases}$$

17. Solve the Laplace equation $\Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0$ in the annulus $1 < r < 2$ (polar coordinates), with boundary values $u(1,\theta) = 0$,

$$u(2,\theta) = 1 - \frac{\theta^2}{\pi^2} \quad \text{for} \quad |\theta| \leq \pi.$$

18. Solve the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = y, & 0 < x < 2,\ 0 < y < 1, \\ u(x,0) = 0, & u(x,1) = 0, \\ u(0,y) = y - y^3, & u(2,y) = 0. \end{cases}$$

19. Solve the problem

$$\begin{cases} \sqrt{1+t}\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 1, & u(1,t) = 0, \\ u(x,0) = 1 - x^2. \end{cases}$$

20. Solve the problem

$$\begin{cases} u''_{xx} + u''_{yy} + 20u = 0, & 0 < x < 1, \quad 0 < y < 1, \\ u(0,y) = u(1,y) = 0, \\ u(x,0) = 0, & u(x,1) = x^2 - x. \end{cases}$$

21. Solve the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = t \sin x, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \sin 2\pi x. \end{cases}$$

22. Solve the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0, \\ u(0,t) = t + 1, \\ u(1,t) = 0, \\ u(x,0) = 1 - x. \end{cases}$$

23. Solve the problem

$$\begin{cases} u_{xx} + 1 = \frac{1}{4} u_{tt}, \qquad 0 < x < 2, \quad t > 0 \\ u(0,t) = 0, \quad u(x,0) = x - x^2, \\ u(2,t) = -2, \quad u_t(x,0) = 0 \end{cases}$$

24. Solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \qquad \sqrt{x^2 + y^2} < 1,$$

with boundary values $f(\theta) = \sin^2 \theta + \cos \theta$ (in polar coordinates).

25. Find a 2π-periodic solution to the equation

$$x'' + 2x' + x = \cos(t).$$

Also find the solution to

$$x'' + 2x' + x = \cos(t), \qquad x(0) = x'(0) = 0$$

and verify that it approaches the periodic solution as t grows.

26. Find a 1-periodic solution to the equation

$$x'' + 2x' + x = \{t\},$$

where $\{t\}$ is the fractional part of t (that is, the 1-periodic function defined by t for $0 \leq t < 1$). Give the answer in real form.

27. The function $f(x)$ is 2-periodic, and $f(x) = (x+1)^2$ for $-1 < x < 1$. Determine a 2π-periodic solution to the equation

$$2y'' - y' - y = f(x).$$

28. The function $f(t)$ is 3-periodic, and

$$f(t) = \begin{cases} t, & 0 \le t \le 1, \\ 1, & 1 < t < 2, \\ 3-t, & 2 \le t \le 3. \end{cases}$$

Determine, as a Fourier series, a periodic solution to

$$y'' + 3y = f(t).$$

29. Show that, in an inner product space, if $u_n \to u$ and $v_n \to v$ (in norm), then $\langle u_n, v_n \rangle \to \langle u, v \rangle$.

30. Show that, if $(e_k)_{k=1}^{\infty}$ is a complete orthonormal system in an inner product space, then

$$\langle f, g \rangle = \sum_{k=1}^{\infty} \langle f, e_k \rangle \langle e_k, g \rangle.$$

31. Find an orthonormal basis for the space of first degree polynomials, with inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx.$$

32. Use the previous exercise to determine the constants a, b that minimize the integral

$$\int_{-1}^{1} |e^x - ax - b|^2\, dx.$$

33. Determine the solution $y(x)$ to $y'' - y = 0$ which minimizes $\displaystyle\int_{-1}^{1} [1 + x - y(x)]^2 dx$.

34. Determine the polynomial $P(x)$ of degree at most 2 that minimizes
(a) $\displaystyle\int_0^{\infty} [\sqrt{x} - P(x)]^2 e^{-x} dx$, (b) $\displaystyle\int_{-\infty}^{\infty} [x^4 - P(x)]^2 e^{-x^2/2} dx$, (c) $\displaystyle\int_0^{\infty} [e^{x/4} - P(x)]^2 x e^{-x} dx$.

35. Determine the polynomial of the form $P(x) = x^3 + ax^2 + bx + c$ that minimizes $\displaystyle\int_0^1 [P(x)]^2 dx$.

36. Let
$$Q_n(x) = \frac{d^n}{dx^n}\left(x^n(1-x)^n\right), \quad n = 0, 1, 2, \dots.$$

(a) Determine the leading term and the constant term in the polynomial $Q_n(x)$.

(b) Show that $Q_n(x)$ and $Q_m(x)$ are orthogonal in $L^2(0,1)$ if $n \neq m$.

(c) Determine the norm $\|Q_n\|$ of $Q_n(x)$ in $L^2(0,1)$.

37. Apply Parseval's formula to the 2π-periodic function $f(x) = x$, $|x| < \pi$. Use the result to compute $\sum_1^\infty \frac{1}{n^2}$.

38. Apply Parseval's formula to the 2π-periodic function $f(x) = x^2$, $|x| < \pi$. Use the result to compute $\sum_1^\infty \frac{1}{n^4}$.

39. Define $J_n(x)$ through the Fourier series
$$e^{ix\sin(t)} = \sum_{n=-\infty}^{\infty} J_n(x)e^{int}$$
(they are called Bessel functions). Compute, for $x \in \mathbb{R}$,
$$\sum_{n=-\infty}^{\infty} |J_n(x)|^2.$$

40. Determine all eigenvalues and eigenfunctions for the Sturm–Liouville problem
$$\begin{cases} f'' + \lambda f = 0, & 0 < x < a, \\ f(0) - f'(0) = 0, & f(a) + 2f'(a) = 0. \end{cases}$$

41. Determine all eigenvalues and eigenfunctions for the Sturm–Liouville problem
$$\begin{cases} -e^{-4x}\dfrac{d}{dx}\left(e^{4x}\dfrac{du}{dx}\right) = \lambda u, & 0 < x < 1, \\ u(0) = 0, \quad u'(1) = 0. \end{cases}$$
Expand the function e^{-2x} as a series in the eigenfunctions.

42. Using the table in Folland's book, compute the Fourier transform of

(a) $x\,\chi_{[-1,1]}(x)$,

(b) $\sin x\,\chi_{[-\pi,\pi]}(x)$,

(c) $e^{-x}H(x)$,

(d) $e^{-|x|}\cos(x)$,

(e) $\dfrac{1}{x^2+6x+13}$,

(f) $\dfrac{x}{(x^2+1)^2}$,

(g) $\dfrac{1}{(t^2+1)^2}$.

Here, H is Heaviside's function and $\chi_{[a,b]}(x)$ is the characteristic function of $[a,b]$, that is, $\chi_{[a,b]}(x) = 1$ for $x \in [a,b]$ and 0 else.

43. If $f(x)$ has Fourier transform $\hat{f}(\xi)$, what is then the Fourier transform of $\cos(x)f(2x+1)$?

44. Complete the proof of (a) and (b) in Folland, Theorem 7.5.

45. Use Fourier transform to find a function f such that

$$\int_{-\infty}^{\infty} f(x-y)e^{-|y|}\,dy = 2e^{-|x|} - e^{-2|x|}.$$

46. For a and b positive, use Fourier transform to compute the integrals

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(at)\sin(b(u-t))}{t(u-t)}\,dt$,

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(at)\sin(bt)}{t^2}\,dt$.

47. Using Exercise 22(e), compute

$$\int_{-\infty}^{\infty} \frac{(x^2+2)^2}{(x^4+4)^2}\,dx.$$

48. The function $f(t)$ has Fourier transform $\hat{f}(\omega) = \frac{\omega}{1+\omega^4}$. Compute

a) $\displaystyle\int_{-\infty}^{\infty} tf(t)dt$, b) $f'(0)$.

49. The function $f(t)$ has Fourier transform $\dfrac{1-i\omega}{1+i\omega}\dfrac{\sin\omega}{\omega}$. Compute $\displaystyle\int_{-\infty}^{\infty} |f(t)|^2 dt$.

50. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\,dx$ using Fourier transform.

51. The function $f(t)$ has Fourier transform $\dfrac{1}{|\omega|^3+1}$. Compute $\displaystyle\int_{-\infty}^{\infty} |f * f'|^2 dt$.

52. Determine the Fourier transform of the function

$$f(t) = \int_0^2 \frac{\sqrt{\omega}}{1+\omega} e^{i\omega t} d\omega.$$

Compute a) $\displaystyle\int_{-\infty}^{\infty} f(t)\cos t dt$,   b) $\displaystyle\int_{-\infty}^{\infty} |f(t)|^2 dt$.

53. Let $f(t) = \displaystyle\int_0^1 \sqrt{\omega}\, e^{\omega^2} \cos \omega t \, d\omega$. Compute $\displaystyle\int_{-\infty}^{\infty} |f'(t)|^2 dt$.

54. The continuous function $f(x)$ has Fourier transform $\widehat{f}(\xi) = \frac{\ln(1+\xi^2)}{\xi^2}$. Determine $f(0)$ and $\int_{-\infty}^{\infty} f(x)\, dx$.

55. Show that the functions $\varphi_n(x) = \dfrac{\sin \frac{x}{2}}{\pi x} e^{inx}$ are pairwise orthogonal in $L^2(\mathbf{R})$. Determine numbers c_n such that

$$\int_{-\infty}^{\infty} \left| \frac{1}{1+x^2} - \sum_{n=-N}^{N} c_n \varphi_n(x) \right|^2 dx$$

is minimal. Is the orthogonal system $(\varphi_n)_{n\in\mathbb{Z}}$ complete?

56. Find (as an integral) a solution to the heat equation $u_{xx} = u_t$ for $t > 0$, $x \in \mathbb{R}$, where $u(x,0) = 1$ for $|x| < 1$ and 0 else.

57. Find a bounded solution to

$$\begin{cases} u_t = k u_{xx}, & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = (1-2x^2)e^{-x^2}, & -\infty < x < \infty. \end{cases}$$

58. Show that, if f is even, then under suitable conditions

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F(\xi) \cos(\xi x)\, d\xi,$$

where

$$F(\xi) = \int_0^{\infty} f(x) \cos(\xi x)\, dx.$$

Also give a corresponding formula for odd functions.

59. Let $f \in L^2(\mathbb{R})$ be given by $f(t) = \sin(at)\sin(bt)/t$, with a and b positive. Show that f is band-limited in the sense that $\hat{f}(\omega) = 0$ for $|\omega| > a + b$. You may use that standard properties of the Fourier transform remain valid for L^2 functions, even though we have not gone into this is detail.

60. If δ and Ω are two numbers with $0 < \pi/\Omega < \delta$, find a function $f \in L^2(\mathbb{R})$, such that $\hat{f}(\omega) = 0$ for $|\omega| \geq \Omega$ and $f(n\delta) = 0$ for $n \in \mathbb{Z}$, but $f \neq 0$ as an element of $L^2(\mathbb{R})$. This shows that the Shannon–Nyquist sampling distance $\Delta t = \pi/\omega_{\max}$ is the largest possible.
Hint: Use the previous Exercise.

61. Determine the finite (discrete) Fourier transform of the sequence $(1, 0, 0, -1)$. Check the inversion formula.

62. Give a version of Plancherel's formula for finite Fourier transform.

63. Let $x(n)$ be N-periodic, and

$$x(n) = \begin{cases} 1, & 0 \leq n \leq k-1, \\ 0, & k \leq n \leq N-1. \end{cases}$$

Compute the discrete Fourier transform. Using Parseval's formula, compute the sum

$$\sum_{\mu=1}^{N-1} \frac{1 - \cos\frac{2\pi\mu k}{N}}{1 - \cos\frac{2\pi\mu}{N}} \, .$$

64. Determine the discrete Fourier transform of the N-periodic function $x(n) = \sin\dfrac{n\pi}{N}, \quad n = 0, \ldots, N-1$.

Answers and hints:

1. (a) **Hint:** Make the change of variables $y = x + ct$, $z = x - ct$.
   (c) $x^2 + 4t^2 + xt$.

2. $u(x,t) = \sin(2x)\cos(2ct) + \dfrac{3}{5c}\sin(5x)\sin(5ct)$.

5. $\dfrac{e^{\cos(4\pi^2)} + e}{2}$.

6. $12\sum_{n=1}^{\infty} \frac{(-1)^n \sin(nx)}{n^3}$. 0 resp $\frac{3}{8}\pi^3$.

7. The Fourier series is
$$\frac{\sinh(\pi)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n e^{inx}}{1+n^2}$$
   and the sum is
$$\frac{\pi}{2}\coth(\pi) + \frac{1}{2}.$$

8. The Fourier series is
$$\frac{a\sin(\pi a)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n e^{inx}}{a^2-n^2}.$$

9. The integral equals $\pi/2$.

10. $\cos x = \dfrac{8}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx \quad (0 < x < \frac{\pi}{2})$. The value of the sum is $\dfrac{\pi^2}{64}$.

11. $\dfrac{1}{2}(e^{-75t}\sin(5x) - e^{-27t}\sin(3x))$.

12. $\dfrac{8}{\pi}\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}e^{-(2k+1)^2 t}\sin((2k+1)x)$.

13. (a) $T - \dfrac{4T}{\pi}\displaystyle\sum_{k=0}^{\infty}\frac{1}{2k+1}e^{-(2k+1)^2\pi^2 at/l^2}\sin((2k+1)\pi x/l)$.
    (b) The cooking time is approximately $\dfrac{l^2}{a\pi^2}\ln\left(\dfrac{16}{3\pi}\right)$, so a twice as thick steak takes four times as long to cook.

14. Second question: In one of the points $a = kl/7$, $k = 1, 2, \ldots, 6$. Note that these are the nodes of the sixth overtone.

15. Second question: Yes, take $\delta = kl/7$, $k = 1, 2, \ldots, 6$.

16. $u(x,t)=\frac{2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^{n-1}}{n^2\pi^2}e^{-\sqrt{\frac{n\pi}{2}}}\cos\left(n\pi t-\sqrt{\frac{n\pi}{2}}x\right)$

17. $\frac{2}{3\ln 2}\ln r+\frac{2}{\pi^2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^{n+1}}{n^2(2^n-2^{-n})}(r^n-r^{-n})e^{in\theta}$

18. $u(x,y)=\frac{1}{6}(y^3-y)+\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\sinh 2n\pi}(\sinh n\pi x+7\sinh n\pi(2-x))\sin n\pi y$

19. $u(x,t)=1-x+\frac{8}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}e^{-\frac{2}{3}(2k+1)^2\pi^2[(1+t)^{3/2}-1]}\sin(2k+1)\pi x$

20. $u(x,y)=-\frac{8}{\pi^3}\sin\pi x\frac{\sin(\sqrt{20-\pi^2}y)}{\sin\sqrt{20-\pi^2}}-$
$-\frac{8}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k+1)^3}\sin(2k+1)\pi x\frac{\sinh(\sqrt{(2k+1)^2\pi^2-20}y)}{\sinh\sqrt{(2k+1)^2\pi^2-20}}$

21. $u(x,t)=e^{-4\pi^2t}\sin 2\pi x+2\pi\sin 1\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2\pi^2-1}\left[\frac{t}{n^2\pi^2}-\frac{1}{n^4\pi^4}(1-e^{-n^2\pi^2t})\right]\sin n\pi x$

22.

$$\begin{aligned}u(x,t)&=(t+1)(1-x)+\sum_{n=1}^{\infty}\frac{1}{n^3\pi^3}(e^{-2n^2\pi^2t}-1)\sin n\pi x=\\&=(t+1)(1-x)+\frac{x^2}{4}-\frac{x}{6}-\frac{x^3}{12}+\sum_{n=1}^{\infty}\frac{1}{n^3\pi^3}e^{-2n^2\pi^2t}\sin n\pi x\end{aligned}$$

23. $u(x,t)=-\frac{x^2}{2}+\frac{16}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\cos((2k+1)\pi t)\sin(\frac{(2k+1)\pi}{2}x)$

24. $u(x,y)=\frac{1}{2}(y^2-x^2)+x+\frac{1}{2}$, or in polar coordinates $u(r,\theta)=-\frac{1}{2}r^2\cos 2\theta+r\cos\theta+\frac{1}{2}$.

25. $x(t)=\frac{1}{2}\sin t$, $x(t)=\frac{1}{2}\sin t-\frac{1}{2}te^{-t}$.

26. $x(t)=\frac{1}{2}-\sum_{n=1}^{\infty}\frac{\sin(2\pi nt-2\arctan(2\pi n))}{\pi n(1+4\pi^2n^2)}$.

27. $f(x)=\frac{4}{3}+\frac{2}{\pi^2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n(1+in\pi)}{n^2}e^{in\pi x},\quad y=-\frac{4}{3}+\frac{2}{\pi^2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^{n-1}(1+in\pi)}{n^2(2n^2\pi^2+in\pi+1)}e^{in\pi x}$

28. $y(t)=\frac{2}{9}-\sum_{n=1}^{\infty}\frac{3(1-\cos\frac{2n\pi}{3})}{\pi^2n^2(3-\frac{4}{9}n^2\pi^2)}\cos\frac{2n\pi t}{3}$

29. **Hint:** Write $\langle u, v\rangle - \langle u_n, v_n\rangle = \langle u, v - v_n\rangle + \langle u - u_n, v_n\rangle$.

30. **Hint:** Use Exercise 15.

31. For instance, $\frac{1}{\sqrt{2}}$ and $\sqrt{\frac{3}{2}}x$.

32. $a = 3e^{-1}$, $b = (e - e^{-1})/2$.

33. $$\frac{2\sinh 1}{\frac{1}{2}\sinh 2 + 1}\cosh x + \frac{2e^{-1}}{\frac{1}{2}\sinh 2 - 1}\sinh x$$

34. (a) $\frac{\sqrt{\pi}}{16}(3 + 6x - \frac{1}{2}x^2)$ (b) $3(2x^2 - 1)$ (c) $\frac{8}{81}(x^2 + 12)$

35. $x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$

36. (a) Leading term $(-1)^n((2n)!/n!)x^n$, constant term $n!$. (b) $n!/\sqrt{2n+1}$.

37. $\pi^2/6$.

38. $\frac{\pi^4}{90}$.

39. 1, for any x.

40. Eigenvalues $\lambda_k = \nu_k^2$, where ν_k are the positive solutions to $\tan \nu a = \frac{3\nu}{2\nu^2 - 1}$.
Eigenfunctions: $\nu_k \cos \nu_k x + \sin \nu_k x$.

41. $\lambda_1 = 4 - \beta_1^2$, where β_1 is the positive root of $\tanh\beta = \frac{\beta}{2}$; $u_1(x) = e^{-2x}\sinh \beta_1 x$
$\lambda_n = 4 + \beta_n^2$, where β_n, $n = 2, 3, \ldots$ are the positive roots of $\tan\beta = \frac{\beta}{2}$; $u_n(x) = e^{-2x}\sin\beta_n x$
$$e^{-2x} = \sum_{n=1}^{\infty} \frac{2\sqrt{\lambda_N}[\sqrt{\lambda_n} + 2(-1)^n]}{\beta_n(\lambda_n - 2)}\,u_n(x)$$

42. (a) $2i\frac{\xi\cos\xi - \sin\xi}{\xi^2}$, (b) $\frac{2i\sin(\pi\xi)}{\xi^2 - 1}$, (c) $\frac{1}{1 + i\xi}$, (d) $\frac{2(\xi^2 + 2)}{\xi^4 + 4}$, (e) $\frac{1}{2}\pi e^{3i\xi - 2|\xi|}$, (f) $-\frac{1}{2}\pi i\xi e^{-|\xi|}$, (g) $\frac{\pi}{2}(1 + |\omega|)e^{-|\omega|}$.

43. $$\frac{1}{4}\left(e^{\frac{1}{2}i(\xi-1)}\hat{f}\left(\frac{\xi - 1}{2}\right) + e^{\frac{1}{2}i(\xi+1)}\hat{f}\left(\frac{\xi + 1}{2}\right)\right).$$

45. $f(x) = \frac{3}{2}e^{-2|x|}$.

46. (a) $\pi \dfrac{\sin(cu)}{u}$, (b) πc, where $c = \min(a, b)$.

47. $3\pi/8$.

48. a) i b) $\dfrac{i}{2\sqrt{2}}$

49. $\dfrac{1}{2}$

50. $\pi(1 - e^{-1})$

51. $\dfrac{1}{9\pi}$

52. $\hat{f}(\omega) = \dfrac{2\pi\sqrt{\omega}}{1+\omega}$ when $0 < \omega < 2$, 0 else. a) $\dfrac{\pi}{2}$, b) $2\pi\left(\ln 3 - \dfrac{2}{3}\right)$

53. $\dfrac{\pi}{8}(e^2 + 1)$

54, $f(0) = 1$, $\int_{-\infty}^{\infty} f(x)dx = 1$.

55. $c_n = \begin{cases} \pi(e^{\frac{1}{2}} - e^{-\frac{1}{2}})e^{-|n|}, & n \neq 0 \\ 2\pi(1 - e^{-\frac{1}{2}}), & n = 0. \end{cases}$
The system is not complete.

56. $u(x,t) = \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{-1}^{1} e^{-(x-y)^2/4t}\, dy.$

57. $u(x,t) = \dfrac{4kt + 1 - 2x^2}{(4kt+1)^{5/2}} e^{-\frac{x^2}{4kt+1}}$

58. See Folland, page 238.

63. The value of the sum is $k(N - k)$.

64. $X(\mu) = \displaystyle\sum_{n=0}^{N-1} x(n) e^{-2\pi i \mu n/N} = \dfrac{\sin \frac{\pi}{N}}{\cos \frac{2\mu\pi}{N} - \cos \frac{\pi}{N}}$