Chalmers University of Technology and Göteborg University
Mathematics
Hjalmar Rosengren August 31, 2010

1 Laplace transform

As an introduction to Fourier methods we discuss Laplace transform. As we will see later, it is a very close relative of Fourier transform. Without going deeply into the theory, we will explain how Laplace transform can be used in practical computation. In particular, it is very useful for solving initial value problems for ordinary differential equations.

1.1 Definition

Let f be a function defined for $t \geq 0$. The Laplace transform of f is defined as

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt, \tag{1}$$

provided that the integral exists. Often, we will write $F = \mathcal{L}(f)$.
Example. The function $f(t) = e^t$ has Laplace transform

$$F(s) = \int_0^\infty e^{(1-s)t}\,dt = \left[\frac{e^{(1-s)t}}{1-s}\right]_{t=0}^\infty = \frac{1}{s-1}, \qquad s > 1.$$

Example. The function $f(t) = e^{t^2}$ does not have a Laplace transform, since the integral

$$\int_0^\infty e^{t^2-st}\,dt$$

is divergent for all s.

In order for the integral (1) to exist, f must satisfy some regularity condition and some growth restriction at infinity. We will assume that the functions we are dealing with are *piecewise continuous*. This means that they are continuous except for isolated jumps; see the Appendix. We will also assume that the functions have *exponential order*. This means that there exist constants C and a so that

$$|f(t)| \leq Ce^{at}, \qquad t \geq 0. \tag{2}$$

Lemma 1.1. *If f is piecewise continuous and of exponential order, then the Laplace transform $F(s)$ exists for $s > a$, where a is any constant such that* (2) *holds.*

Proof. Piecewise continuity implies that the integral exists over bounded intervals, so we only need to worry about convergence when the upper end-point tends to ∞. Using (2), we have

$$\int_0^\infty |f(t)e^{-st}|\,dt \leq C\int_0^\infty e^{(a-s)t}\,dt < \infty, \qquad s > a,$$

so under our assumptions the integral is absolutely convergent. ∎

Although we mostly think of s as a real variable, it is sometimes important to consider complex values of s. Then, the inequality $s > a$ should be replaced by $\mathrm{Re}(s) > a$.

Usually, one should avoid memorizing things, but the following table of Laplace transforms may be an exception.

$f(t)$	t^k	e^{at}	$\sin(at)$	$\cos(at)$
$F(s)$	$\frac{k!}{s^{k+1}}$	$\frac{1}{s-a}$	$\frac{a}{a^2+s^2}$	$\frac{s}{a^2+s^2}$

Exercise 1. Verify all entries in the table above. For what values of s do the Laplace transforms exist?

Exercise 2. Define the function f by $f(t) = 1$ for all $t \neq 1$, but $f(1) = 2$. What is $\mathcal{L}(f)$?

1.2 Properties

We will give some important properties of Laplace transform. The first is linearity:

$$\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g),$$

where a and b are scalars. This is immediate from linearity of the integral.

The second is the *shift rule*: If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(e^{ct}f(t)) = F(s-c)$. (The kind of notation used here is somewhat sloppy but completely standard.) The proof is simple:

$$\int_0^\infty e^{ct} f(t) e^{-st}\, dt = \int_0^\infty f(t) e^{-(s-c)t}\, dt.$$

For the next property we need to be more precise about the assumptions.

Lemma 1.2. *Suppose f is continuous, piecewise C^1 (see the Appendix) and of exponential order. Let $F = \mathcal{L}(f)$ and let a be such that* (2) *holds. Then, for $s > a$,*

$$\mathcal{L}(f'(t)) = sF(s) - f(0).$$

Proof. By partial integration (which can be used even though f' may have jumps, see the Appendix)

$$\int_0^R f'(t)e^{-st}\, dt = \left[f(t)e^{-st}\right]_{t=0}^R + s\int_0^R f(t)e^{-st}\, dt = f(R)e^{-sR} - f(0) + s\int_0^R f(t)e^{-st}\, dt.$$

Let $R \to \infty$. By definition, the left-hand side tends to $\mathcal{L}(f'(t))$. By (2), the first term on the right tends to 0 if $s > a$, and the last term to $sF(s)$. $\square$

Thus, Laplace transform converts differentiation to multiplication by s (and adding a boundary term). This is the basis for its usefulness for differential equations.

Iterating Lemma 1.2 gives that, under natural assumptions on f,

$$\mathcal{L}(f^{(n)}(t)) = s^n F(s) - (s^{n-1} f(0) + s^{n-2} f'(0) + \cdots + f^{(n-1)}(0)). \tag{3}$$

.

The final property that we mention is that, if $\mathcal{L}(f) = F$, then

$$\mathcal{L}(tf(t)) = -F'(s)$$

and, by iteration,

$$\mathcal{L}(t^k f(t)) = (-1)^k F^{(k)}(s).$$

Formally, this can be proved by differentiating under the integral sign:

$$F'(s) = \frac{d}{ds} \int_0^\infty f(t) e^{-st}\, dt = \int_0^\infty \frac{d}{ds} f(t) e^{-st}\, dt = -\int_0^\infty t f(t) e^{-st}\, dt = -\mathcal{L}(tf(t)).$$

Is this allowed? You may have learned that

$$\frac{d}{ds} \int_0^\infty f(t,s)\, dt = \int_0^\infty \frac{\partial}{\partial s} f(t,s)\, dt$$

if one can estimate

$$|f(t,s)| \leq A(t), \qquad |f'_s(t,s)| \leq B(t),$$

where

$$\int_0^\infty A(t) dt < \infty, \qquad \int_0^\infty B(t) dt < \infty.$$

In our case, if we assume (2) and choose $s > a$, then we may use the obvious estimates

$$|f(t) e^{-st}| \leq e^{(a-s)t}, \qquad |tf(t) e^{-st}| \leq t e^{(a-s)t}.$$

We summarize our main findings in a table.

$f(t)$	$e^{ct} f(t)$	$f'(t)$	$f^{(n)}(t)$	$tf(t)$
$F(s)$	$F(s-c)$	$sF(s) - f(0)$	$s^n F(s) - \sum_{j=1}^n s^{n-j} f^{(j-1)}(0)$	$-F'(s)$

Exercise 3. Compute the Laplace transform of the following functions:

(a) $\sinh(t)$,

(b) $t^2 \sin(3t)$,

(c) $\cos^2 t$.

Exercise 4. If $\mathcal{L}(f(t)) = F(s)$, what is $\mathcal{L}(f(at))$?

1.3 Convolution

As we will see, the concept of *convolution* is quite important in Fourier analysis. When f and g are functions of a real variable, their convolution (Swedish: faltning) is defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\,ds.$$

We can also write this as a line integral

$$\int_{x+y=t} f(x)g(y)\,d\sigma,$$

which makes it obvious that

$$f * g = g * f.$$

One can verify that

$$(f * g) * h = f * (g * h),$$

for instance, by showing that both sides equal the surface integral

$$\iint_{x+y+z=t} f(x)g(y)h(z)\,d\Sigma.$$

See also Folland, Section 7.1.

In the context of Laplace transform, we work with functions defined for $t \geq 0$. Given two such functions f, g, extend them to the real line by defining $f(t) = g(t) = 0$ for $t < 0$ (such functions are often called *causal*). Then, their convolution is given by

$$(f * g)(t) = \int_{0}^{t} f(s)g(t-s)\,ds.$$

Proposition 1.3. *Let f and g be causal, piecewise continuous and of exponential order. Then,*

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g). \tag{4}$$

Proof. We write

$$\mathcal{L}(f)(s)\mathcal{L}(g)(s) = \int_0^{\infty} e^{-sx} f(x)\,dx \int_0^{\infty} e^{-sy} g(y)\,dy = \iint_{x,y\geq 0} e^{-s(x+y)} f(x)g(y)\,dxdy.$$

We now integrate over line segments $x + y = t$, $x, y \geq 0$ (draw a picture!). If you are comfortable with this, just write the integral as

$$\int_0^{\infty} e^{-st} \left(\int_{x+y=t} f(x)g(y)\,d\sigma \right) dt = \mathcal{L}(f * g)(s).$$

Otherwise, the change of variables $x + y = t$, $y = u$ (which has Jacobian 1) gives

$$\int_0^{\infty} e^{-st} \left(\int_0^{t} f(t-u)g(u)\,du \right) dt = \mathcal{L}(f * g)(s).$$

The assumption of exponential order ensures that all integrals are absolutely convergent for s big enough. This makes the calculations rigorous. □

Exercise 5. Let $f(t) = t$ and $g(t) = \cos(t)$ for $t \geq 0$, and consider them to vanish for $t < 0$. Compute the convolution $f * g$ and check that $\mathcal{L}(f)\mathcal{L}(g) = \mathcal{L}(f * g)$.

1.4 Uniqueness

If f and g have the same Laplace transform is then $f = g$?

The answer is no if the question is interpreted completely literally (cf. Exercise 2), but the following theorem shows that one can usually pretend that the answer is yes.

Theorem 1.4 (Uniqueness Theorem)**.** *If f and g are piecewise continuous and of exponential order and $\mathcal{L}(f) = \mathcal{L}(g)$, then $f(t) = g(t)$ for all points $t \geq 0$ where both functions are continuous.*

This is not so easy to prove. We will give a proof much later during the course, using Fourier transform.

When $\mathcal{L}(f) = F$, one often writes $f = \mathcal{L}^{-1}(F)$. However, one should keep in mind that f is only determined by F outside its jump points, so the inverse Laplace transform $\mathcal{L}^{-1}$ is not quite well-defined.

2 Applications

2.1 Initial value problems

Example. Use Laplace transform to solve the initial value problem

$$f''(t) - 3f'(t) + 2f(t) = t \qquad (t > 0), \qquad f(0) = 0, \quad f'(0) = 1. \tag{5}$$

We assume that f is of exponential order; in particular, it has a Laplace transform F. Using (3), we get

$$\big(s^2F(s) - sf(0) - f'(0)\big) - 3\big(sF(s) - f(0)\big) + 2F(s) = \frac{1}{s^2}.$$

Plugging in the initial values and simplifying we have

$$(s^2 - 3s + 2)F(s) - 1 = \frac{1}{s^2} \quad \Longleftrightarrow \quad F(s) = \frac{1 + s^2}{s^2(s-1)(s-2)}.$$

The next step is to make a partial fraction decomposition. I assume you have learned this as part of the technique for integrating rational functions. In the case at hand, we get

$$F(s) = \frac{3}{4s} + \frac{1}{2s^2} - \frac{2}{s-1} + \frac{5}{4(s-2)}.$$

Looking in our table, we see that

$$f(t) = \frac{3}{4} + \frac{1}{2}\,t - 2e^t + \frac{5}{4}\,e^{2t} \tag{6}$$

has Laplace transform $F(s)$. By Theorem 1.4, this must be the solution f that we seek.

Maybe you noted that, although the method of solution is straight-forward, there are logical gaps in the argument. We have showed that *if* (5) has a solution of exponential order, then it is given by (6). However, could it be that (5) has *no* solution of exponential order so that (6) is *not* a solution? Or could it be that (5) has *many* solutions, but only one of them is of exponential order and can be obtained by our method?

To give a satisfactory answer to these questions one needs to know something about ODEs. For instance, by the general theory of linear ODEs, any initial value problem

$$a_n(t)f^{(n)}(t) + \cdots + a_0(t)f(t) = b(t), \qquad f(0) = c_0, \quad \ldots, \quad f^{(n-1)}(0) = c_{n-1}$$

has a unique solution near $t = 0$, provided that all the functions a_i and b are continuous, and $a_n(0) \neq 0$. In the special case of equations with *constant coefficients*, any solution of (5) has the form $f(t) = p(t) + Ae^{\lambda_1 t} + Be^{\lambda_2 t}$, where p is a first degree polynomial and $\lambda_i = 1, 2$ are the roots of the characteristic polynomial $\lambda^2 - 3\lambda + 2$. In particular, all solutions have exponential order. Thus, (6) is in fact the unique solution of (5). This argument works for any linear ODE with constant coefficients, but for variable coefficients one has to be more careful (see Exercise 10). Throughout the course, we will use Fourier methods to solve differential equations. However, we often ignore many technical points, like questions of uniqueness. You have to learn about such things in courses on differential equations.

Finally, we remark that (6) solves (5) not only for $t > 0$, but in fact for all t. But note that if we apply the same method to the problem

$$f''(t) - 3f'(t) + 2f(t) = |t|, \qquad f(0) = 0, \quad f'(0) = 1, \tag{7}$$

then we obtain the same answer (6) (since t and $|t|$ have the same Laplace transform). In this case, the solution is *not* valid for $t < 0$.

Exercise 6. Compute "the" inverse Laplace transform of the following functions:

(a) $\dfrac{1}{s^2 + 10s + 21}$,

(b) $\dfrac{s + 8}{s^2 + 4s + 5}$,

(c) $\dfrac{s - 1}{s^2 + 2s + 5}$.

Why are there quotation marks on "the"?

Exercise 7. Find a function with Laplace transform $1/s^2(s^2 + 1)$ in two ways. First, do a partial fraction decomposition. Second, use (4).

Exercise 8. Solve, using Laplace transform, the initial value problems

(a) $x'' + x' - 2x = 5e^{-t}\sin(t), \qquad x(0) = 1, \quad x'(0) = 0,$

(b) $x'' + 8x' + 16x = 16\sin(4t), \qquad x(0) = -\frac{1}{2}, \quad x'(0) = 1.$

Exercise 9. Solve, using Laplace transform, the system

$$\begin{cases} 3x' + 3y' - 2x = e^t, \\ x' + 2y' - y = 1, \end{cases}$$

with initial conditions $x(0) = y(0) = 1$.

Exercise 10.

(a) Apply Laplace transform to the equation

$$t^2 u''(t) - 2u(t) = 0, \qquad t > 0.$$

Use this to find the one-parameter family of solutions $u(t) = At^2$.

(b) Rewrite the equation in terms of the function $v(x) = u(e^x)$. Deduce that the general solution is $u(t) = At^2 + Bt^{-1}$.

(c) Why does the method of Laplace transform not give the general solution? You should be able to answer part (c) without going through the calculations leading to (a) and (b).

Exercise 11. Solve the equation (7) for all t.

2.2 A few words on systems and signals

Laplace transform is widely used by engineers working with e.g. signal processing. Even if you are not an engineer, it is useful to have an idea of this viewpoint, and we will return to it throughout the course.

Basically, in engineering a "system" means something that transforms an "input signal" u to an "output signal" x. For instance, think of an electronic circuit with resistors, capacitors etc. The input can then be the voltage in a variable power source, and the output the current measured in some other part of the circuit. Many systems can be modeled by linear ODEs

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_0 x = u.$$

If a_i are constants, the system is called time-invariant.

It is very common to consider input signals such as triangular waves or square waves. For such right-hand sides, the elementary Ansatz method for solving ODEs is tedious. With Laplace transform, computations can be kept to a minimum.

2.3 Piecewise defined functions

In order to compute with functions such as square waves, it is useful to express them in terms of Heaviside's function

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}$$

The value of $H(0)$ is usually chosen as 1, 1/2, 0 or is left un-defined. Since we eventually want to put H into integrals, the choice seldom matters.

Please draw the graphs of a function $f(t)$, the function $f(t)H(t-a)$ and the function $f(t-a)H(t-a)$. Note that $f(t)H(t-a)$ can be thought of as "activating" f at time $t = a$. To get the function $f(t-a)H(t-a)$ we take the graph for $t > 0$ and translate it to the interval $t > a$. Keep these pictures in mind.

For $a \geq 0$, there is a simple rule for the Laplace transform of $f(t-a)H(t-a)$. Indeed,

$$\int_0^\infty f(t-a)H(t-a)e^{-st}\,ds = \int_a^\infty f(t-a)e^{-st}\,ds = \int_0^\infty f(t)e^{-s(t+a)}\,ds = e^{-as}F(s).$$

This is the "second shift rule"

$$\boxed{\begin{array}{ll} f(t) & f(t-a)H(t-a)\ (a \geq 0) \\ F(s) & \qquad e^{-as}F(s) \end{array}}\,.$$

Example: Calculate the Laplace transform of the function

$$f(t) = \begin{cases} 0, & t < 1, \\ (t-1)^2, & 1 \leq t < 2, \\ 1, & t \geq 2. \end{cases}$$

Of course, it is not hard to compute

$$\int_0^\infty f(t)e^{-st}\,dt = \int_1^2 (t-1)^2 e^{-st}\,dt + \int_2^\infty e^{-st}\,dt$$

directly, but we illustrate a somewhat more systematic way. We first express f as

$$f(t) = (t-1)^2 H(t-1) + \left(1 - (t-1)^2\right) H(t-2).$$

For instance, the second term can be read as "at time $t = 2$ activate the function 1 and turn off the function $(t-1)^2$". We then write

$$1 - (t-1)^2 = 2t - t^2 = -(t-2)^2 - 2(t-2),$$

which gives

$$f(t) = (t-1)^2 H(t-1) - (t-2)^2 H(t-2) - 2(t-2)H(t-2)$$

and, by the second shift rule,

$$F(s) = \frac{2}{s^3}\,e^{-s} - \frac{2}{s^3}\,e^{-2s} - \frac{2}{s^2}\,e^{-2s}.$$

Note that, even though you may dismiss f as an "un-natural" function, F certainly seems very natural!

Example: Solve the differential equation

$$x'' + 3x' + 2x = f(t), \qquad x(0) = x'(0) = 0, \qquad t \geq 0,$$

where $f(t) = 1$ for $0 \leq t < 1$ and 0 else.

Since $t \geq 0$ we can write

$$f(t) = 1 - H(t-1).$$

Thus, we obtain

$$\begin{aligned}
&(s^2 + 3s + 2)X(s) = \frac{1 - e^{-s}}{s} \\
\iff\quad & X(s) = \frac{1 - e^{-s}}{s(s+1)(s+2)} \\
\iff\quad & X(s) = (1 - e^{-s})\left(\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right) \\
\iff\quad & x(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - H(t-1)\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right) \\
&\quad = \begin{cases} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}, & 0 \leq t < 1, \\ (e-1)e^{-t} + \frac{1}{2}(1 - e^2)e^{-2t}, & t \geq 1. \end{cases}
\end{aligned}$$

Exercise 12. Let f be the function in the example above. Compute $f * f$ in two ways, first directly and then using (4). If you can, give a probabilistic interpretation of the answer.

Exercise 13. Let

$$g(t) = \begin{cases} t, & 0 < t < 1, \\ 2 - t, & 1 < t < 2, \\ 0, & t > 2. \end{cases}$$

Solve the initial value problem

$$x'' + 2x' + x = g(t), \qquad x(0) = x'(0) = 0.$$

Exercise 14. Let $\{t\}$ denote the fractional part of t, e.g. $\{\pi\} = 0.141592\ldots$. Compute the Laplace transform of $\{t\}$.

2.4 Impulse response

Suppose we replace the square pulse f in the preceding example with the pulse

$$f_\varepsilon(t) = \begin{cases} \dfrac{1}{\varepsilon}, & 0 \le t < \varepsilon, \\ 0, & t > \varepsilon \end{cases}$$

(make a picture!). What happens when $\varepsilon \to 0$?

Note that the area under the graph is constant equal to 1. If we think of f_ε as a force, then this area is *impulse*, that is, the change of momentum caused by the force. The limiting case $\varepsilon \to 0$ corresponds to the impulse 1 applied momentarily at time $t = 0$. One can think of the limit as a "function" δ which satisfies

$$\delta(t) = \begin{cases} \infty, & t = 0, \\ 0, & t \neq 0, \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

No such function exists, but nobody can forbid us to think about it! It is usually called the Dirac delta function. One can give meaning to δ within the framework of *distributions*, but we will not go into that. We take the more pedestrian view-point that any statement about δ is an abbreviation for a statement about the functions f_ε as $\varepsilon \to 0$.

Let F_ε denote the Laplace transform of f_ε. We have

$$f_\varepsilon(t) = \frac{1}{\varepsilon} - \frac{1}{\varepsilon} H(t - \varepsilon),$$

$$F_\varepsilon(s) = \frac{1}{\varepsilon s} - \frac{e^{-\varepsilon s}}{\varepsilon s} = \frac{1 - e^{-\varepsilon s}}{\varepsilon s} \to 1 \qquad (\varepsilon \to 0).$$

This suggests that one should put

$$\mathcal{L}(\delta) = 1. \tag{8}$$

Computing formally, we now consider the system

$$\begin{cases} a_n x^{(n)} + \cdots + a_0 x = \delta, \\ x(0) = x'(0) = \cdots = x^{(n-1)}(0) = 0. \end{cases}$$

That is, we start from a system at rest and then apply an impulse at time $t = 0$. Taking Laplace transform, using (8), gives

$$X(s) = \frac{1}{a_n s^n + \cdots + a_0}.$$

Expanding $X(s)$ as a partial fraction, one can in principle compute the inverse Laplace transform $x(t)$. It is called the *impulse response* (by engineers) or the *fundamental solution* (by mathematicians).

Alternatively, the impulse response can be introduced without invoking the delta "function" as follows.

Proposition 2.1. *The impulse response, as defined above, can also be defined as the solution to*

$$\begin{cases} a_n x^{(n)} + \cdots + a_0 x = 0, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad x^{(n-1)}(0) = \dfrac{1}{a_n}. \end{cases}$$

Proof. Taking Laplace transform, the initial condition $x^{(n-1)}(0) = 1/a_n$ gives rise to a single non-zero term and we get

$$a_n \left(s^n X(s) - \frac{1}{a_n} \right) + a_{n-1} s^{n-1} X(s) + \cdots + a_0 X(s) = 0 \qquad \Longrightarrow \qquad X(s) = \frac{1}{a_n s^n + \cdots + a_0}.$$

□

Exercise 15. Using Laplace transform, solve the problem

$$x' - x = f_\varepsilon(x), \qquad x(0) = 0,$$

where f_ε is as above. Denoting the solution by x_ε, compute $\lim_{\varepsilon \to 0} x_\varepsilon$ and show that it agrees with the impulse response.

2.5 The general inhomogeneous problem

Now look at the problem

$$\begin{cases} a_n x^{(n)} + \cdots + a_0 x = u, \\ x(0) = x'(0) = \cdots = x^{(n-1)}(0) = 0, \end{cases} \tag{9}$$

where u is continuous and of exponential order. Moreover, let h denote the impulse response and $H(s) = 1/(a_n s^n + \cdots + a_0)$ its Laplace transform. We then have

$$(a_n s^n + \cdots + a_0) X(s) = U(s)$$

or

$$X(s) = H(s) U(s).$$

Since multiplication of Laplace transforms corresponds to convolution, this gives the integral formula

$$x(t) = (h * u)(t) = \int_0^t h(s) u(t-s)\, ds \tag{10}$$

for the solution to (9). This result is of great theoretical importance. It is the simplest instance of the method of Green's function, which appears in much more general contexts. We remark that (10) is in fact valid for any continuous u (even when the Laplace transform of u does not exist), as one can verify directly.

Exercise 16. Solve the equation

$$x'' - 3x' + 2x = e^t, \qquad x(0) = x'(0) = 0,$$

using Laplace transform. Verify that the same answer is obtained from (10).

Exercise 17. Verify directly that (10) solves (9).

Appendix: Piecewise continuous and piecewise C^1 functions.

A function f is said to be *piecewise continuous* if it is continuous, except for a finite number of discontinuities in any bounded interval. At those points, we do not even require that f is defined. However, as t approaches a discontinuity point, $f(t)$ must have finite left-hand and right-hand limits (that is, only jump discontinuities are allowed).

A function f is said to be *piecewise* C^1 if it has a continuous derivative except at a finite number of points in any bounded interval. Moreover, as t approaches any such point, both $f(t)$ and $f'(t)$ have finite left-hand and right-hand limits. Note that if f is piecewise C^1, then f' is piecewise continuous.

Suppose that f is continuous and piecewise C^1 (so the graph can have corners but no jumps). Then, the fundamental theorem of calculus

$$\int_a^b f'(t)dt = f(b) - f(a)$$

holds. To understand why, consider the case when f' is continuous in the interval except at one point c. Then,

$$\int_a^b f'(t)\,dt = \int_a^c f'(t)\,dt + \int_c^b f'(t)\,dt = (f(c) - f(a)) + (f(b) - f(c)) = f(b) - f(a). \quad (11)$$

(Here, we use that f does not jump at $t = c$.) In the general case there will be more terms, but contributions from the interior points cancel.

Replacing f by fg in (11), we obtain the partial integration formula

$$\int_a^b f'(t)g(t)\,dt = \big[f(t)g(t)\big]_a^b - \int_a^b f(t)g'(t)\,dt$$

for f and g are continuous and piecewise C^1.

Exercise 18. Which functions are piecewise continuous? Which are piecewise C^1?

(a) $f(x) = \begin{cases} \sin(1/x), & x > 0, \\ 0, & x \le 0, \end{cases}$

(b) $f(x) = \begin{cases} x\sin(1/x), & x > 0, \\ 0, & x \le 0, \end{cases}$

(c) $f(x) = \sqrt[3]{x}$,

(d) $f(x) = \begin{cases} x\{1/x\}, & x > 0, \\ 0, & x \le 0, \end{cases}$

where $\{\cdot\}$ is as in Exercise 14.

Exercise 19. Generalize the fundamental theorem of calculus to functions that are piecewise C^1 but not necessarily continuous.

Answers (may contain some mistakes!)

(2) $1/s$.

(3a) $1/(s^2-1)$.

(3b) $18(s^2-3)/(s^2+9)^3$.

(3c) $(s^2+2)/s(s^2+4)$.

(4) $F(s/a)/a$.

(5) $1-\cos t$.

(6a) $(e^{-3t}-e^{-7t})/4\cdot$

(6b) $e^{-2t}(\cos t+6\sin t)$.

(6c) $e^{-t}(\cos 2t-\sin 2t)$.

(7) $t-\sin t$.

(8a) $e^t-\dfrac{1}{2}\,e^{-2t}+\dfrac{1}{2}\,e^{-t}(\cos t-3\sin t)$.

(8b) $te^{-4t}-\dfrac{1}{2}\,\cos 4t$.

(9) $x(t)=\dfrac{3}{2}\,e^{t/3}-\dfrac{1}{2}\,e^t$, $y(t)=-1+\dfrac{3}{2}\,e^{t/3}+\dfrac{1}{2}\,e^t$.

(11) $\begin{cases} \frac{3}{4}+\frac{t}{2}-2e^t+\frac{5}{4}e^{2t}, & t>0, \\ -\frac{3}{4}-\frac{t}{2}+\frac{3}{4}e^{2t}, & t<0. \end{cases}$

(12) $\begin{cases} t, & 0<t\le 1, \\ 2-t, & 1<t<2, \\ 0, & \text{else.} \end{cases}$

(13) $x(t)=f(t)-2H(t-1)f(t-1)+H(t-2)f(t-2)$, where $f(t)=(t+2)e^{-t}+t-2$.

(14) $(1+s-e^s)/s^2(1-e^s)$.

(16) $x(t)=e^{2t}-(t+1)e^t$.

(18) (b) and (c) are piecewise continuous; none of the functions piecewise C^1.

(19) $\displaystyle\int_a^b f'(x)dx = f(b_-)-f(a_+)+\sum_x (f(x_-)-f(x_+))$, where the sum is over all x in $a<x<b$ where f is discontinuous.