Fourier analysis (MMG710/TMA362)

Time: 2009-10-24, 08.30–13.30
Tools: No calculator or handbook is allowed
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Grades: Each problem gives 4 points. For MMG710 grades are G (12-17 points) and VG (18-24 points). For TMA362 grades are 3 (12-14 points), 4 (15-17 points) and 5 (18-24 points).

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1 Use Laplace transform to solve the initial value problem

$$x''(t) - 3x'(t) + 2x(t) = e^t, \qquad x(0) = x'(0) = 0.$$

2 Using that $1/(x^2 + a^2)$ has Fourier transform $\pi e^{-a|\xi|}/a$, compute the integral

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2(x^2+4)^2}\, dx.$$

3 Solve (as an infinite series) the problem

$$\begin{cases} u_t' = 2u_{xx}'', & t > 0, \quad 0 < x < \pi, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = 1, & 0 < x < \pi. \end{cases}$$

4 Prove that if $(e_k)_{k=1}^N$ are orthonormal vectors in an inner product space, then

$$\left\| u - \sum_{k=1}^{N} a_k e_k \right\|$$

assumes its minimum when $a_k = \langle u, e_k \rangle$.

5 (a) Let $\phi_n(x)$ denote the function that equals 1 for $x \in [n - \frac{1}{2}, n + \frac{1}{2}]$ and 0 otherwise. Compute the Fourier transform $\hat{\phi}_n(\xi)$.

(b) Prove that $(\hat{\phi}_n)_{n=-\infty}^{\infty}$ is an orthogonal system in $L^2(\mathbb{R})$. Prove that it is not complete.

6 (a) Show that

$$\frac{\sin x}{x} = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos(nx), \qquad 0 < x < \pi,$$

where

$$b_n = \frac{1}{\pi} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin x}{x}\, dx.$$

(b) Use the result above to compute

$$\int_0^{\infty} \frac{\sin x}{x}\, dx.$$

Good luck! Hjalmar