Fourier analysis (MMG710/TMA362)

Time: 2010-10-23, 08.30–13.30
Tools: No calculator or handbook is allowed.
Questions: Peter Helgesson, 0703-088304.
Grades: Each problem gives a maximum of 4 points. For MMG710 grades are G (12-17 points) and VG (18-24 points). For TMA362 grades are 3 (12-14 points), 4 (15-17 points) and 5 (18-24 points).

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1 Expand the function $f(x) = x$ as a Fourier cosine series in the interval $0 < x < \pi$. Compute the value of this cosine series at the point $x = 7$.

2 If $f(t)$ has Laplace transform $F(s)$, which functions have Laplace transform $F(s-a)$ and $e^{-as}F(s)$, where $a \geq 0$? Prove your statements.

3 Solve the problem

$$\begin{cases} u'_t = 3u''_{xx}, & t > 0, \quad 0 < x < \pi, \\ u'_x(0,t) = u'_x(\pi,t) = 0, & \\ u(x,0) = \sin^2 x. & \end{cases}$$

4 Let $f(x) = \sin(\pi x)/(x^2 - 1)$.

(a) Show that

$$\hat{f}(\xi) = \begin{cases} \pi i \sin(\xi), & |\xi| \leq \pi, \\ 0, & |\xi| > \pi. \end{cases}$$

(b) Compute

$$\int_{-\infty}^{\infty} \frac{\sin^2(\pi x)}{(x^2-1)^2}\, dx.$$

5 Define the notion of convergence in the space $L^2([0,1])$. Show that if a sequence of functions converges uniformly to 0 in $[0,1]$, then it converges to 0 in $L^2([0,1])$.

6 A metal thread is bent into a circle. The ends are attached so that they are partially, but not completely, insulated from each other. The corresponding heat transfer problem can be modeled by

$$u'_t = ku''_{xx}, \qquad 0 < x < 1, \quad t > 0, \qquad u_x(0) = u_x(1) = \alpha\big(u(0) - u(1)\big),$$

where α and k are positive constants. Looking for separated solutions, $u(x,t) = X(x)T(t)$, one finds (you do not need to do this) that X satisfies the Sturm–Liouville problem

$$X''(x) + \lambda X(x) = 0, \qquad X'(0) = X'(1) = \alpha\big(X(0) - X(1)\big).$$

(a) Prove that the problem is symmetric in the sense that $\langle f'', g\rangle_{L^2([0,1])} = \langle f, g''\rangle_{L^2([0,1])}$, when f and g are sufficiently differentiable and satisfy the boundary conditions.

(b) Prove that the problem has a non-trivial solution when $\lambda = 4n^2\pi^2$, $n \in \mathbb{Z}$. Prove that there are infinitely many other values of λ for which the problem also has a non-trivial solution.

Good luck! Hjalmar