Fourier analysis (MMG710/TMA362)

Time: 2011-01-08, 08.30–13.30
Tools: No calculator or handbook is allowed.
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Grades: Each problem gives a maximum of 4 points. For MMG710 grades are G (12-17 points) and VG (18-24 points). For TMA362 grades are 3 (12-14 points), 4 (15-17 points) and 5 (18-24 points).

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1 Derive an identity for the Laplace transform of a convolution of two functions.

2 Show that $e^{-|x|}$ has Fourier transform $2/(\xi^2+1)$. Use this result to compute the Fourier transform of $1/(x^2+2x+3)$.

3 Solve the inhomogeneous problem

$$\begin{cases} u'_t = 2u''_{xx} + \cos x, & 0 < x < \pi, \quad t > 0 \\ u'_x(0,t) = u'_x(\pi,t) = 0, & t > 0, \\ u(x,0) = 1, & 0 < x < \pi. \end{cases}$$

4 In standard tables, one can find the identity

$$\sum_{n=1}^{\infty} \frac{\sin(na)\sin(nx)}{n^2} = \begin{cases} x(\pi-a)/2, & 0 < x < a, \\ a(\pi-x)/2, & a < x < \pi, \end{cases}$$

where $0 < a < \pi$ (you do not have to prove this). Use this result to sum the series

$$\sum_{n=1}^{\infty} \frac{\sin^2(na)}{n^2}$$

and

$$\sum_{n=1}^{\infty} \frac{\sin^2(na)}{n^4}.$$

5 Formulate and prove a theorem on uniform convergence of Fourier series.

6 Let $\phi_n(x) = \sin(nx)$ for $0 < x < \pi$ and $\phi_n(x) = 0$ else. Compute values of c_n which minimize the integral

$$\int_{-\infty}^{\infty} \left| \frac{\sin\xi}{\xi} - \sum_{n=1}^{\infty} c_n \hat{\phi}_n(\xi) \right|^2 d\xi.$$

Also compute the minimum value. You may use that $\sin(\xi)/\xi = \hat{\psi}(\xi)$, where $\psi(x) = 1/2$ for $|x| < 1$ and $\psi(x) = 0$ else.

Good luck! Hjalmar