Fourier analysis (MMG710/TMA362)

Time: 2011-08-15, 08.30–13.30
Tools: No calculator or handbook is allowed.
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Grades: Each problem gives 4 points. For MMG710 grades are G (12-17 points) and VG (18-24 points). For TMA362 grades are 3 (12-14 points), 4 (15-17 points) and 5 (18-24 points).

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1 Using the method of Fourier, solve the boundary value problem

$$\begin{aligned} u'_t &= u''_{xx}, \\ u'_x(0,t) &= u'_x(\pi,t) = 0, \\ u(x,0) &= \cos\left(\frac{x}{2}\right), \end{aligned}$$

in the region $t > 0$, $0 < x < \pi$.

2 Let

$$f(t) = \begin{cases} t, & 0 < t < 2, \\ 2, & t > 2. \end{cases}$$

Using Laplace transform, solve the initial value problem

$$x''(t) + x(t) = f(t), \qquad x(0) = 0, \quad x'(0) = 1.$$

3 Define the function χ by $\chi(x) = 1$ for $|x| < 1$ and $\chi(x) = 0$ for $|x| > 1$. Using that $\hat{\chi}(\xi) = 2\sin(\xi)/\xi$, compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin(\xi - 1)}{\xi - 1} \frac{\sin(2\xi)}{2\xi}\, d\xi.$$

4 Suppose that you have some orthonormal vectors $e_1, \ldots, e_N$ in an inner product space, and want to approximate some vector u with a linear combination $a_1e_1 + \cdots + a_Ne_N$. How can you find the "best" coefficients a_k? Prove that the method that you propose really gives the best approximation (in a sense that you have to formulate precisely).

5 (a) Formulate and prove a theorem on differentiation of Fourier series.

(b) Determine all 2π-periodic and twice continuously differentiable functions u such that $u''(x) = u(x + \pi)$ for all x.

6 Consider the Sturm–Liouville problem

$$f'' + \lambda f = 0, \qquad f'(0) = f(0), \quad f'(1) = \beta f(1).$$

For which values of β does there exist an eigenvalue λ in the interval $0 \le \lambda \le \pi^2/4$?

Good luck! Hjalmar