Fourier analysis (MMG710/TMA362)

Time: 2011-10-22, 08.30–12.30
Tools: Only the attached sheet of formulas. No calculator or handbook is allowed.
Questions: Hossein Raufi, 0703-088304.
Grades: Each problem gives 4 points. For MMG710 grades are G (12-17 points) and VG (18-24 points). For TMA362 grades are 3 (12-14 points), 4 (15-17 points) and 5 (18-24 points).

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1 Expand the function $f(x) = e^x$ as a Fourier sine series on the interval $0 < x < 1$. What is the sum of the series at the point $x = 3/2$?

2 Let $0 < a < \pi$.

(a) Prove that

$$\frac{a}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(na)\cos(nx)}{n} = \begin{cases} 1, & 0 < x < a, \\ 0, & a < x < \pi. \end{cases}$$

(b) Use the result above to compute

$$\sum_{n=1}^{\infty} \frac{\sin^2(na)}{n^2}.$$

3 Recall that a function f is called *band-limited* if $\hat{f}$ vanishes outside some bounded interval.

(a) Show that the convolution of two band-limited functions is band-limited.

(b) Show that the product of two band-limited functions is band-limited.

4 (a) Formulate a uniqueness theorem for the Laplace transform. Write down two functions with the same Laplace transform and explain why this does not contradict the theorem.

(b) Prove the theorem you stated above, using the inversion theorem for Fourier transform.

5 Let

$$f(t) = \int_0^1 \sqrt{\xi} e^{i\xi t}\, d\xi.$$

Compute

$$\int_{-\infty}^{\infty} |f'(t)|^2\, dt.$$

6 Solve the problem (for $t > 0$, $0 < x < \pi$)

$$\begin{cases} u'_t = (t+1)u''_{xx}, \\ u(0,t) = 0, \\ u(\pi,t) = 1, \\ u(x,0) = 0. \end{cases}$$

Good luck! Hjalmar