Fourier analysis (MMG710/TMA362)

Time: 2012-01-07, 08:30–12:30
Tools: Only the attached sheet of formulas. No calculator or handbook is allowed.
Questions: Ragnar Freij, 0703-088304.
Grades: Each problem gives 4 points. For MMG710 grades are G (12-17 points) and VG (18-24 points). For TMA362 grades are 3 (12-14 points), 4 (15-17 points) and 5 (18-24 points).

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1 Use Fourier transform to compute the integral

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+2x+2)^2}\, dx.$$

2 Solve the boundary value problem

$$u'_t = 2u''_{xx}, \qquad t > 0, \quad 0 < x < \pi,$$

$$u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = \cos(3x).$$

3 Let

$$f(t) = \begin{cases} t, & 0 < t < 1, \\ 1, & t > 1. \end{cases}$$

Use Laplace transform to solve the initial value problem

$$y' + 2y = f, \qquad y(0) = 0.$$

4 Formulate and prove Bessel's inequality (any version is fine).

5 Let c_n be the coefficients in the Fourier series

$$e^{x^2} = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad 0 < x < 2\pi.$$

Is it true or false that

$$2xe^{x^2} = \sum_{n=-\infty}^{\infty} inc_n e^{inx}, \qquad 0 < x < 2\pi?$$

Motivate your answer carefully. If you use a known theorem, formulate it precisely and explain why all conditions in the theorem hold.

6 Find the complex Fourier series of the function $(\cos x)^n$, where n is a positive integer. Use the result to compute the sum

$$\sum_{k=0}^{n} \binom{n}{k}^2$$

(where $\binom{n}{k} = n!/k!\,(n-k)!$).

Good luck! Hjalmar

Some formulas in Fourier analysis

Trigonometric identities

$$e^{ix} = \cos x + i\sin x, \qquad \cos x = \frac{e^{ix}+e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix}-e^{-ix}}{2i},$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y, \qquad \sin(x+y) = \sin x\cos y + \cos x\sin y,$$

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2},$$

$$\sin x\,\sin y = \frac{\cos(x-y)-\cos(x+y)}{2}, \qquad \cos x\,\cos y = \frac{\cos(x-y)+\cos(x+y)}{2},$$

$$\sin x\,\cos y = \frac{\sin(x+y)+\sin(x-y)}{2}.$$

Hyperbolic functions

$$\cosh x = \frac{e^x+e^{-x}}{2}, \qquad \sinh x = \frac{e^x-e^{-x}}{2}.$$

Laplace transforms

$f(t)$	t^k	e^{at}	$\sin(at)$	$\cos(at)$
$F(s)$	$\frac{k!}{s^{k+1}}$	$\frac{1}{s-a}$	$\frac{a}{a^2+s^2}$	$\frac{s}{a^2+s^2}$

$f(t)$	$f(at)$	$e^{ct}f(t)$	$f(t-a)H(t-a)$	$f^{(n)}(t)$	$tf(t)$
$F(s)$	$F(s/a)/a$	$F(s-c)$	$e^{-as}F(s)$	$s^nF(s)-\sum_{j=1}^n s^{n-j}f^{(j-1)}(0)$	$-F'(s)$

Fourier transforms

$f(x)$	$e^{-x^2/2}$	$\frac{1}{x^2+1}$	$e^{-\lvert x\rvert}$	$\chi(x)$	$\frac{\sin x}{x}$
$\hat{f}(\xi)$	$\sqrt{2\pi}\,e^{-\xi^2/2}$	$\pi e^{-\lvert\xi\rvert}$	$\frac{2}{\xi^2+1}$	$\frac{2\sin\xi}{\xi}$	$\pi\chi(\xi)$

(where $\chi(x)$ equals 1 for $|x|<1$ and 0 else)

$f(x)$	$f(x-c)$	$e^{icx}f(x)$	$f(ax)$	$f'(x)$	$xf(x)$	$\hat{f}(x)$
$\hat{f}(\xi)$	$e^{-ic\xi}\hat{f}(\xi)$	$\hat{f}(\xi-c)$	$\hat{f}(\xi/a)/a$	$i\xi\hat{f}(\xi)$	$i(\hat{f})'(\xi)$	$2\pi f(-\xi)$