Fourier analysis (MMG710/TMA362)

Time: 2012-10-27, 8:30–12:30.
Tools: Only the attached sheet of formulas. No calculator or handbook is allowed.
Questions: Oskar Hamlet, 0703-088304.
Grades: Unless otherwise indicated, each problem gives 4 points. For MMG710 grades are G (12-17 points) and VG (18-24). For TMA362 grades are 3 (12-14 points), 4 (15-17) and 5 (18-24).

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1 For t a real parameter, compute

$$\int_{-\infty}^{\infty} \frac{\cos(tx)}{x^2+4x+7}\,dx.$$

2 Using Laplace transform, solve the initial value problem

$$x'' + x = \begin{cases} t, & 0 < t < 2, \\ 2, & t > 2, \end{cases}$$

with $x(0) = 0$, $x'(0) = 1$.

3 Solve the problem

$$u_t = 3u_{xx}, \qquad t > 0, \quad 0 < x < \pi,$$
$$u_x(0,t) = u_x(\pi,t) = 0, \qquad u(x,0) = e^x.$$

4 Using the identity

$$x(\pi - x) = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}, \qquad 0 < x < \pi$$

(which you do not need to prove), compute

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(8n-4)}{(2n-1)^3}$, (2p)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(2n-1)}{(2n-1)^4}$. (2p)

5 Prove a theorem on uniform convergence of Fourier series.

6 (a) Prove that the identity

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right), \qquad u,\, v \in \mathcal{H}$$

holds in any Hilbert space $\mathcal{H}$. (2p)

(b) Using (a), prove that $\mathrm{L}^1([0,1])$, with the norm

$$\|f\| = \int_0^1 |f(x)|\,dx,$$

is not a Hilbert space. (2p)