Chalmers tekniska högskola och Göteborgs universitet
Matematik
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Gibbs phenomenon

We have seen (Folland, Thm. 2.5) that if f is 2π-periodic, piecewise C^1 and continuous, then its Fourier series converges *uniformly* on $\mathbb{R}$. If, on the other hand, f is *not* continuous, convergence cannot be uniform. This follows from the fact that if a sequence of continuous functions (e.g. the partial sums $\sum_{-N}^{N} c_n e^{inx}$) converges uniformly, then the limit function is continuous. It turns out that for Fourier series the situation is even worse. The partial sums develop "spikes" close to each jump point, whose heights remain positive as $N \to \infty$ (the limit height of the largest spike is about 9% of the height of the jump). This fact is known as the Gibbs phenomenon. See Folland, Figure 2.8 for an illustration.

We will first illustrate the Gibbs phenomenon by an explicit example, and then sketch how the general case actually follows from that example. Consider the 2π-periodic function s defined by $s(x) = \pi - x$ for $0 < x < 2\pi$. It has a jump of height 2π at $x = 0$. Its Fourier series is

$$2\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}.$$

Thus, the error in the Nth Fourier approximation, for $0 < x < \pi$, is

$$g_N(x) = 2\sum_{n=1}^{N} \frac{\sin(nx)}{n} - (\pi - x).$$

Since we are interested in the maximum error, we compute the derivative

$$g_N'(x) = 1 + 2\sum_{n=1}^{N} \cos(nx).$$

We recognize this as the Dirichlet kernel, which we have seen can be written

$$g_N'(x) = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}.$$

From Figure 2.8 it seems that the error g_N is maximal at its smallest positive critical point, that is, at $x_N = \pi/(N+\frac{1}{2})$. If we can show that the error at this point remains positive in the limit $N \to \infty$, that is,

$$\lim_{N\to\infty} g_N(x_N) > 0, \qquad (1)$$

then we can conclude that the Gibbs phenomenon holds for the function s.

We prove (1) using Riemann sums. Namely, we can write

$$g_N(x_N) = 2\sum_{n=1}^{N} \frac{\sin(nx_N)}{n} - (\pi - x_N) = 2\sum_{n=1}^{N} \frac{\sin(\xi_n)}{\xi_n}\,\Delta x - (\pi - x_N),$$

where $\xi_n = nx_N = n\pi/(N+1/2)$ are points at distance $\Delta x = x_N = \pi/(N+1/2)$ in the interval $0 < x < \pi$. By known facts on Riemann sums,

$$\lim_{N\to\infty} g_N(x_N) = 2\int_0^\pi \frac{\sin x}{x}\,dx - \pi.$$

This is approximately 0.562, in particular it is positive. A slightly different proof is indicated in Folland, Exercise 2.6.1.

We now know that the Gibbs phenomenon holds for the function s. This can be used to prove it for *any* piecewise C^1 but discontinuous f. We sketch how this can be done. Note first that $\frac{h}{2\pi}\,s(x-a)$ has a jump of height h at $x = a$. Suppose that the jumps of f in $0 \leq x < 2\pi$ are $h_1, \dots, h_n$ at the points $a_1, \dots, a_n$. Then,

$$g(x) = f(x) - \sum_{j=1}^{n} \frac{h_j}{2\pi} s(x - a_j) \tag{2}$$

is piecewise C^1 and continuous, so its Fourier series converges uniformly on $\mathbb{R}$ (Folland, Thm. 2.5). On the other hand, close to a jump point a_k, one can show that the kth term in the sum (2) exhibits the Gibbs phenomenon, whereas all the other terms do not (the first statement follows from what we did above, but the second statement needs a little work). The Gibbs phenomenon for the kth term must then be cancelled by a corresponding Gibbs phenomenon for f. Thus, f exhibits the Gibbs phenomenon at each jump point.

Exercise: Show that, in the notation used above,

$$\lim_{N\to\infty} g_N\left(\frac{k\pi}{N+\frac{1}{2}}\right) = 2\int_0^{k\pi} \frac{\sin x}{x}\,dx - \pi.$$

Exercise: It is known that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

How can this result be interpreted, in view of the previous exercise?