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Mathematics
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# 1 Laplace transform

The Laplace transform is a close relative of the Fourier transform. It is particularly useful for studying initial value problems, or other systems defined naturally on a half-line $t \geq 0$.

## 1.1 Definition

Let $f$ be a function defined for $t \geq 0$. The Laplace transform of $f$ is defined as

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt,$$

provided that the integral exists. Often, we will write $F = \mathcal{L}(f)$.

*Example.* The function $f(t) = e^t$ has Laplace transform

$$F(s) = \int_0^\infty e^{(1-s)t}\,dt = \left[\frac{e^{(1-s)t}}{1-s}\right]_{t=0}^{\infty} = \frac{1}{s-1}, \qquad s > 1.$$

*Example.* The function $f(t) = e^{t^2}$ does not have a Laplace transform, since the integral

$$\int_0^\infty e^{t^2-st}\,dt$$

is divergent for all $s$.

In order for the Laplace transform to exist, $f$ must satisfy some regularity condition and some growth restriction at infinity. We will assume that the functions we are dealing with are *piecewise continuous*. We will also assume that the functions have *exponential order*. This means that there exist constants $C$ and $a$ so that

$$|f(t)| \leq Ce^{at}, \qquad t \geq 0. \tag{1}$$

**Lemma 1.1.** *If $f$ is piecewise continuous and of exponential order, then the Laplace transform $F(s)$ exists for $s > a$, where $a$ is any constant such that* (1) *holds.*

*Proof.* Piecewise continuity implies that the integral exists over bounded intervals, so we only need to consider convergence when the upper end-point tends to $\infty$. By (1),

$$\int_0^\infty |f(t)e^{-st}|\,dt \leq C\int_0^\infty e^{(a-s)t}\,dt < \infty, \qquad s > a,$$

so under our assumptions the integral is absolutely convergent. $\square$

Although we mostly think of $s$ as a real variable, it is sometimes important to consider complex values of $s$. Then, the inequality $s > a$ should be replaced by $\mathrm{Re}(s) > a$.

In the following table, we give some of the most useful Laplace transforms.

| | | | | |
|---|---|---|---|---|
| $f(t)$ | $t^k$ | $e^{at}$ | $\sin(at)$ | $\cos(at)$ |
| $F(s)$ | $\dfrac{k!}{s^{k+1}}$ | $\dfrac{1}{s-a}$ | $\dfrac{a}{a^2+s^2}$ | $\dfrac{s}{a^2+s^2}$ |

*Exercise* 1. Verify all entries in the table above. For what values of $s$ do the Laplace transforms exist?

*Exercise* 2. Define the function $f$ by $f(t) = 1$ for all $t \neq 1$, but $f(1) = 2$. What is $\mathcal{L}(f)$?

## 1.2 Properties

We will give some important properties of Laplace transform. The first is linearity:

$$\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g),$$

where $a$ and $b$ are scalars. This is immediate from linearity of the integral.

The second is the *shift rule*: If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(e^{ct}f(t)) = F(s-c)$. (The notation used here is a bit sloppy but completely standard.) The proof is simple:

$$\int_0^\infty e^{ct} f(t) e^{-st}\, dt = \int_0^\infty f(t) e^{-(s-c)t}\, dt.$$

For the next property we need to be more precise about the assumptions.

**Lemma 1.2.** *Suppose $f$ is continuous, piecewise $C^1$ and of exponential order. Let $F = \mathcal{L}(f)$ and let $a$ be such that* (1) *holds. Then, for $s > a$,*

$$\mathcal{L}(f'(t)) = sF(s) - f(0).$$

*Proof.* By partial integration (which works even though $f'$ may have jumps)

$$\int_0^R f'(t)e^{-st}\, dt = \left[f(t)e^{-st}\right]_{t=0}^R + s\int_0^R f(t)e^{-st}\, dt = f(R)e^{-sR} - f(0) + s\int_0^R f(t)e^{-st}\, dt.$$

Let $R \to \infty$. By definition, the left-hand side tends to $\mathcal{L}(f'(t))$. By (1), the first term on the right tends to 0 if $s > a$, and the last term to $sF(s)$. $\square$

Thus, Laplace transform converts differentiation to multiplication by $s$ (and adding a boundary term). This is the basis for its usefulness for differential equations.

Iterating Lemma 1.2 gives that, under natural assumptions on $f$,

$$\mathcal{L}(f^{(n)}(t)) = s^n F(s) - (s^{n-1}f(0) + s^{n-2}f'(0) + \cdots + f^{(n-1)}(0)). \tag{2}$$

.

The final property that we mention is that, if $\mathcal{L}(f) = F$, then

$$\mathcal{L}(tf(t)) = -F'(s)$$

and, by iteration,

$$\mathcal{L}(t^k f(t)) = (-1)^k F^{(k)}(s).$$

Formally, this can be proved by differentiating under the integral sign:

$$F'(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty \frac{d}{ds} f(t)e^{-st}\,dt = -\int_0^\infty tf(t)e^{-st}\,dt = -\mathcal{L}(tf(t)).$$

Is this allowed? You may have learned that

$$\frac{d}{ds}\int_0^\infty f(t,s)\,dt = \int_0^\infty \frac{\partial}{\partial s} f(t,s)\,dt$$

if one can estimate

$$|f(t,s)| \leq A(t), \qquad |f'_s(t,s)| \leq B(t),$$

where

$$\int_0^\infty A(t)dt < \infty, \qquad \int_0^\infty B(t)dt < \infty.$$

In our case, if we assume (1) and choose $s > a$, then we may use the obvious estimates

$$|f(t)e^{-st}| \leq e^{(a-s)t}, \qquad |tf(t)e^{-st}| \leq te^{(a-s)t}.$$

We summarize our main findings in a table.

| | | | | |
|---|---|---|---|---|
| $f(t)$ | $e^{ct}f(t)$ | $f'(t)$ | $f^{(n)}(t)$ | $tf(t)$ |
| $F(s)$ | $F(s-c)$ | $sF(s) - f(0)$ | $s^nF(s) - \sum_{j=1}^n s^{n-j}f^{(j-1)}(0)$ | $-F'(s)$ |

*Exercise* 3. Compute the Laplace transform of the following functions:

(a) $\sinh(t)$, (b) $t^2\sin(3t)$, (c) $\cos^2 t$.

*Exercise* 4. If $\mathcal{L}(f(t)) = F(s)$, what is $\mathcal{L}(f(at))$?

## 1.3 Convolution

Recall that, when $f$ and $g$ are functions of a real variable, their convolution (Swedish: faltning) is defined by

$$(f * g)(t) = \int_{-\infty}^\infty f(s)g(t-s)\,ds.$$

In the context of Laplace transform, we work with functions defined for $t \geq 0$. Given two such functions $f$, $g$, extend them to the real line by defining $f(t) = g(t) = 0$ for $t < 0$ (such functions are often called *causal*). Then, their convolution is given by

$$(f * g)(t) = \int_0^t f(s)g(t-s)\,ds.$$

**Proposition 1.3.** *Let $f$ and $g$ be causal, piecewise $C^1$ and of exponential order. Then,*

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g). \tag{3}$$

*Proof.* We have

$$\mathcal{L}(f * g)(s) = \int_0^\infty e^{-st} \left( \int_0^t f(t-u)g(u)\,du \right) dt = \iint_{0 \le u \le t} e^{-st} f(t-u)g(u)\,dt\,du.$$

Making the change of variables $t = u + v$, this integral becomes

$$\iint_{u,\,v \ge 0} e^{-s(u+v)} f(v)g(u)\,du\,dv = \int_0^\infty e^{-sv} f(v)\,dv \int_0^\infty e^{-su} g(u)\,du = \mathcal{L}(f)(s)\mathcal{L}(g)(s).$$

The assumption of exponential order ensures that all integrals are absolutely convergent for $s$ big enough, which makes the calculation rigorous. □

*Exercise* 5. Let $f(t) = t$ and $g(t) = \cos(t)$ for $t \ge 0$, and consider them to vanish for $t < 0$. Compute the convolution $f * g$ and check that $\mathcal{L}(f)\mathcal{L}(g) = \mathcal{L}(f * g)$.

## 1.4 Uniqueness

If $f$ and $g$ have the same Laplace transform, is then $f = g$? The answer is no if the question is interpreted completely literally (cf. Exercise 2), but in practice one can usually pretend that it is yes.

**Theorem 1.4** (Uniqueness Theorem)**.** *If $f$ and $g$ are piecewise $C^1$ and of exponential order and $\mathcal{L}(f) = \mathcal{L}(g)$, then $f(t) = g(t)$ for all points $t \ge 0$ where both functions are continuous.*

This is not easy to prove from scratch. We will give a proof using Fourier transform.

*Proof.* For $b$ a constant, define $h(t) = e^{-bt}f(t)$ for $t \ge 0$ and $h(t) = 0$ for $t < 0$. If $b > a$, with $a$ as in (1), then $h \in L^1(\mathbb{R})$ and

$$\hat{h}(\xi) = \int_{-\infty}^\infty h(t)e^{-it\xi}\,dt = \int_0^\infty e^{-bt} f(t) e^{-it\xi}\,dt = (\mathcal{L}(f))(b + i\xi).$$

By the inversion formula for Fourier transform, $h$ is determined by $\hat{h}$ except for the values at jump points. Thus, $f$ is determined by $\mathcal{L}(f)$, again except for values at jumps. □

More explicitly, the inversion formula for Fourier transform gives

$$f(t) = e^{bt}h(t) = \lim_{r\to\infty} \frac{e^{bt}}{2\pi} \int_{-r}^r \hat{h}(\xi)e^{it\xi}\,d\xi = \lim_{r\to\infty} \frac{1}{2\pi} \int_{-r}^r (\mathcal{L}(f))(b+i\xi)e^{(b+i\xi)t}\,d\xi$$

if $f$ is continuous at $t$ and $b$ is large enough. This tells us how $f$ can be recovered from $\mathcal{L}(f)$, but in practice a table (or mathematical software) is more useful.

Theorem 1.4 also holds for functions that are just piecewise continuous instead of piecewise $C^1$. This follows similarly, using the theorem on p. 218 in Folland (which we skipped).

When $\mathcal{L}(f) = F$, one often writes $f = \mathcal{L}^{-1}(F)$. However, one should keep in mind that $f$ is only determined by $F$ outside its jump points, so the inverse Laplace transform $\mathcal{L}^{-1}$ is not quite well-defined.

# 2 Applications

## 2.1 Initial value problems

*Example.* Use Laplace transform to solve the initial value problem

$$f''(t) - 3f'(t) + 2f(t) = t \qquad (t > 0), \qquad f(0) = 0, \quad f'(0) = 1. \tag{4}$$

Suppose that $f$ is of exponential order. Then, it has a Laplace transform $F$. By (2),

$$\big(s^2F(s) - sf(0) - f'(0)\big) - 3\big(sF(s) - f(0)\big) + 2F(s) = \frac{1}{s^2}.$$

Plugging in the initial values and simplifying we have

$$(s^2 - 3s + 2)F(s) - 1 = \frac{1}{s^2} \quad\iff\quad F(s) = \frac{1 + s^2}{s^2(s-1)(s-2)}.$$

The next step is to make a partial fraction decomposition. I assume you have learned this as part of the technique for integrating rational functions. In the case at hand, we get

$$F(s) = \frac{3}{4s} + \frac{1}{2s^2} - \frac{2}{s-1} + \frac{5}{4(s-2)}.$$

Looking in our table, we see that

$$f(t) = \frac{3}{4} + \frac{1}{2}\,t - 2e^t + \frac{5}{4}\,e^{2t} \tag{5}$$

has Laplace transform $F(s)$. By Theorem 1.4, this must be the solution $f$ that we seek.

Maybe you noted that, although the method of solution is straight-forward, there are logical gaps in the argument. We have showed that *if* (4) has a solution of exponential order, then it is given by (5). However, could it be that (4) has *no* solution of exponential order so that (5) is *not* a solution? Or could it be that (4) has *many* solutions, but only one of them is of exponential order and can be obtained by our method?

To give a satisfactory answer to these questions one needs to know something about ODEs. By the general theory of linear ODEs, any initial value problem

$$a_n(t)f^{(n)}(t) + \cdots + a_0(t)f(t) = b(t), \qquad f(0) = c_0, \quad \ldots, \quad f^{(n-1)}(0) = c_{n-1}$$

has a unique solution near $t = 0$, provided that the functions $a_i$ and $b$ are continuous, and $a_n(0) \neq 0$. For equations with *constant coefficients* such as (4), there are solution methods that tell us what the solution looks like, if the right-hand side is nice. For instance, in the case of (4) the solution must have the form $f(t) = p(t) + Ae^{\lambda_1 t} + Be^{\lambda_2 t}$, where $p$ is a first degree polynomial and $\lambda_i$ are the roots of the characteristic polynomial $\lambda^2 - 3\lambda + 2$. In particular, the solution has exponential order. Thus, (5) is in fact the unique solution of (4). For variable coefficients one must be more careful (see Exercise 10).

Finally, we remark that (5) solves (4) not only for $t > 0$, but in fact for all $t$. But note that if we apply the same method to the problem

$$f''(t) - 3f'(t) + 2f(t) = |t|, \qquad f(0) = 0, \quad f'(0) = 1, \tag{6}$$

then we obtain the same answer (5) (since $t$ and $|t|$ have the same Laplace transform). In this case, the solution is *not* valid for $t < 0$.

*Exercise* 6. Compute "the" inverse Laplace transform of the following functions:

$$\text{(a)}\ \frac{1}{s^2 + 10s + 21}, \qquad \text{(b)}\ \frac{s-1}{s^2 + 2s + 5}, \qquad \text{(c)}\ \frac{s}{(s^2+1)^2}, \qquad \text{(d)}\ \frac{1}{(s^2+1)^2}.$$

Why are there quotation marks on "the"?

*Exercise* 7. Solve, using Laplace transform, the initial value problems

(a) $x'' + 3x' + 2x = e^t, \qquad x(0) = 1, \quad x'(0) = 1,$

(b) $x'' + 8x' + 16x = 16\sin(4t), \qquad x(0) = -\frac{1}{2}, \quad x'(0) = 1,$

(c) $x'' + x' - 2x = 5e^{-t}\sin(t), \qquad x(0) = 1, \quad x'(0) = 0,$

(d) $x^{(3)} - 6x'' + 11x' - 6x = 1, \qquad x(0) = x'(0) = x''(0) = 0.$

*Exercise* 8. Solve, using Laplace transform, the system

$$\begin{cases} 3x' + 3y' - 2x = e^t, \\ x' + 2y' - y = 1, \end{cases}$$

with initial conditions $x(0) = y(0) = 1$.

*Exercise* 9.

(a) Apply Laplace transform to the equation

$$t^2 u''(t) - 2u(t) = 0, \qquad t > 0.$$

Use this to find the one-parameter family of solutions $u(t) = At^2$.

(b) Rewrite the equation in terms of the function $v(x) = u(e^x)$. Deduce that the general solution is $u(t) = At^2 + Bt^{-1}$.

(c) Why does the method of Laplace transform not give the general solution? You should be able to answer part (c) without going through the calculations leading to (a) and (b).

## 2.2 A few words on systems and signals

Laplace transform is widely used by engineers working with e.g. signal processing. Even if you are not an engineer, it is useful to have an idea of this viewpoint.

Basically, in engineering a "system" means something that transforms an "input signal" $u$ to an "output signal" $x$. For instance, think of an electronic circuit with resistors, capacitors etc. The input could be the voltage in a variable power source, and the output the current measured in some other part of the circuit. Many systems can be modeled by linear ODEs

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_0 x = u.$$

If $a_i$ are constants, the system is called time-invariant.

It is very common to consider input signals such as triangular waves or square waves. For such right-hand sides, the elementary Ansatz method for solving ODEs is tedious. With Laplace transform, computations can be kept to a minimum.

## 2.3 Piecewise defined functions

In order to compute with functions such as square waves, it is useful to express them in terms of Heaviside's function

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}$$

The value of $H(0)$ is usually chosen as 1, 1/2, 0 or is left un-defined. Since we eventually want to put $H$ into integrals, the choice does not matter to us.

Please draw the graphs of a function $f(t)$, the function $f(t)H(t-a)$ and the function $f(t-a)H(t-a)$. Note that $f(t)H(t-a)$ can be thought of as "activating" $f$ at time $t = a$. To get the function $f(t-a)H(t-a)$ we take the graph for $t > 0$ and translate it to the interval $t > a$. Keep these pictures in mind.

For $a \geq 0$, there is a simple rule for the Laplace transform of $f(t-a)H(t-a)$. Indeed,

$$\int_0^\infty f(t-a)H(t-a)e^{-st}\,ds = \int_a^\infty f(t-a)e^{-st}\,ds = \int_0^\infty f(t)e^{-s(t+a)}\,ds = e^{-as}F(s).$$

This is the "second shift rule"

$$\boxed{\begin{array}{ll} f(t) & f(t-a)H(t-a)\ (a \geq 0) \\ F(s) & e^{-as}F(s) \end{array}}\,.$$

**Example:** Calculate the Laplace transform of the function

$$f(t) = \begin{cases} 0, & t < 1, \\ (t-1)^2, & 1 \leq t < 2, \\ 1, & t \geq 2. \end{cases}$$

Of course, it is not hard to compute

$$\int_0^\infty f(t)e^{-st}\,dt = \int_1^2 (t-1)^2 e^{-st}\,dt + \int_2^\infty e^{-st}\,dt$$

directly, but we illustrate a more systematic way. We first express $f$ as

$$f(t) = (t-1)^2 H(t-1) + \left(1-(t-1)^2\right) H(t-2).$$

For instance, the second term can be read as "at time $t=2$ activate the function 1 and turn off the function $(t-1)^2$". We then write

$$1-(t-1)^2 = 1-((t-2)+1)^2 = -(t-2)^2 - 2(t-2),$$

which gives

$$f(t) = (t-1)^2 H(t-1) - (t-2)^2 H(t-2) - 2(t-2)H(t-2)$$

and, by the second shift rule,

$$F(s) = \frac{2}{s^3}\,e^{-s} - \frac{2}{s^3}\,e^{-2s} - \frac{2}{s^2}\,e^{-2s}.$$

**Example:** Solve the differential equation

$$x'' + 3x' + 2x = f(t), \qquad x(0) = x'(0) = 0, \qquad t \geq 0,$$

where $f(t) = 1$ for $0 \leq t < 1$ and 0 else.

Since $t \geq 0$ we can write

$$f(t) = 1 - H(t-1).$$

Thus, we obtain

$$\begin{aligned}
& (s^2+3s+2)X(s) = \frac{1-e^{-s}}{s} \\
\iff\quad & X(s) = \frac{1-e^{-s}}{s(s+1)(s+2)} \\
\iff\quad & X(s) = (1-e^{-s})\left(\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right) \\
\iff\quad & x(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - H(t-1)\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right) \\
& \quad = \begin{cases} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}, & 0 \leq t < 1, \\ (e-1)e^{-t} + \frac{1}{2}(1-e^2)e^{-2t}, & t \geq 1. \end{cases}
\end{aligned}$$

**Example:** Let $f(t)$ be the square wave defined as 1 in the intervals $0 < t < 1$, $2 < t < 4$, $4 < t < 6$, ... and as 0 else. Solve the initial value problem

$$x'(t) + x(t) = f(t), \quad t \geq 0, \qquad x(0) = 0.$$

(By definition, the solution is the unique continuous function so that the differential equation holds outside the jump points.)

Since there are infinitely many jumps, we rewrite $f$ as an infinite sum

$$f(t) = 1 - H(t-1) + H(t-2) - H(t-3) + \cdots .$$

So

$$F(s) = \frac{1}{s}\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots\right) = \frac{1}{s(1+e^{-s})} \qquad (s > 0),$$

by the formula for a geometric series. Taking Laplace transform of the equation gives

$$(s+1)X(s) = \frac{1}{s(1+e^{-s})} \qquad \Longrightarrow \qquad X(s) = \left(\frac{1}{s} - \frac{1}{s+1}\right)\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots\right).$$

Let $g(t) = 1 - e^{-t}$ so that $G(s) = 1/s - 1/(s+1)$. Then,

$$x(t) = g(t) - g(t-1)H(t-1) + g(t-2)H(t-2) - g(t-3)H(t-3) + \cdots .$$

This answer may be good enough, but we can write it more explicitly. If $n < t < n+1$ for some integer $n$, then only the first $n+1$ terms contribute and we have

$$x(t) = \sum_{j=0}^{n}(-1)^j g(t-j) = \sum_{j=0}^{n}(-1)^j(1-e^{j-t}) = \begin{cases} -\dfrac{1-e^{n+1}}{1+e}\, e^{-t}, & n \text{ odd}, \\ 1 - \dfrac{1+e^{n+1}}{1+e}\, e^{-t}, & n \text{ even}. \end{cases} \tag{7}$$

*Exercise* 10. Let $f(t) = 1$ for $0 \le t < 1$ and 0 else. Compute $f * f$ in two ways, first directly and then using (3). If you can, give a probabilistic interpretation of the answer.

*Exercise* 11. Let

$$g(t) = \begin{cases} t, & 0 < t < 1, \\ 2-t, & 1 < t < 2, \\ 0, & t > 2. \end{cases}$$

Solve the initial value problem

$$x'' + 2x' + x = g(t), \qquad x(0) = x'(0) = 0.$$

*Exercise* 12. A beam is clamped at the left end ($x = 0$) and free at the right end ($x = 1$). A load is distributed uniformly along the left half of the beam. In appropriate units, the deflection $y(x)$ of the beam is described by the boundary value problem

$$y^{(4)}(x) = H\left(x - \frac{1}{2}\right) - H(x) = \begin{cases} -1, & 0 < x < 1/2, \\ 0, & 1/2 < x < 1, \end{cases}$$

$$y(0) = y'(0) = y''(1) = y^{(3)}(1) = 0.$$

Determine $y(x)$ using Laplace transform.

*Exercise* 13. Let $\{t\}$ denote the fractional part of $t$, e.g. $\{\pi\} = 0.141592\ldots$. Compute the Laplace transform of $\{t\}$.

*Exercise* 14. Verify that the last equation in (7) is correct, and that the function $x$ defined in this way is continuous.

## 2.4 Impulse response

Let's consider an initial value problem with right-hand side

$$f_\varepsilon(t) = \begin{cases} \dfrac{1}{\varepsilon}, & 0 \le t < \varepsilon, \\ 0, & t > \varepsilon \end{cases}$$

(make a picture!). What happens when $\varepsilon \to 0$?

Note that the area under the graph is constant equal to 1. If we think of $f_\varepsilon$ as a force, then this area is *impulse*, that is, the change of momentum caused by the force. The limiting case $\varepsilon \to 0$ corresponds to the impulse 1 applied momentarily at time $t = 0$. One can think of the limit as a "function" $\delta$ which satisfies

$$\delta(t) = \begin{cases} \infty, & t = 0, \\ 0, & t \neq 0, \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

No such function exists, but nobody can forbid us to think about it! It is usually called the Dirac delta function. One can give meaning to $\delta$ within the framework of *distributions*, but we will not go into that. We take the more pedestrian view-point that any statement about $\delta$ is an abbreviation for a statement about the functions $f_\varepsilon$ as $\varepsilon \to 0$.

Let $F_\varepsilon$ denote the Laplace transform of $f_\varepsilon$. We have

$$f_\varepsilon(t) = \frac{1}{\varepsilon} - \frac{1}{\varepsilon} H(t - \varepsilon),$$

$$F_\varepsilon(s) = \frac{1}{\varepsilon s} - \frac{e^{-\varepsilon s}}{\varepsilon s} = \frac{1 - e^{-\varepsilon s}}{\varepsilon s} \to 1 \qquad (\varepsilon \to 0).$$

This suggests that one should put

$$\mathcal{L}(\delta) = 1. \tag{8}$$

Computing formally, we now consider the system

$$\begin{cases} a_n x^{(n)} + \cdots + a_0 x = \delta, \\ x(0) = x'(0) = \cdots = x^{(n-1)}(0) = 0. \end{cases}$$

That is, we start from a system at rest and then apply an impulse at time $t = 0$. Taking Laplace transform, using (8), gives

$$X(s) = \frac{1}{a_n s^n + \cdots + a_0}.$$

Expanding $X(s)$ as a partial fraction, one can in principle compute the inverse Laplace transform $x(t)$. It is called the *impulse response* (by engineers) or the *fundamental solution* (by mathematicians).

Alternatively, the impulse response can be introduced without invoking the delta "function" as follows.

**Proposition 2.1.** *The impulse response, as defined above, can also be defined as the solution to*

$$\begin{cases} a_n x^{(n)} + \cdots + a_0 x = 0, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad x^{(n-1)}(0) = \dfrac{1}{a_n}. \end{cases}$$

*Proof.* Taking Laplace transform, the initial condition $x^{(n-1)}(0) = 1/a_n$ gives rise to a single non-zero term and we get

$$a_n \left( s^n X(s) - \frac{1}{a_n} \right) + a_{n-1} s^{n-1} X(s) + \cdots + a_0 X(s) = 0 \qquad \Longrightarrow \qquad X(s) = \frac{1}{a_n s^n + \cdots + a_0}.$$

□

*Exercise* 15. Using Laplace transform, solve the problem

$$x' - x = f_\varepsilon(t), \qquad x(0) = 0,$$

where $f_\varepsilon$ is as above. Denoting the solution by $x_\varepsilon$, compute $\lim_{\varepsilon \to 0} x_\varepsilon$ and show that it agrees with the impulse response.

## 2.5 The general inhomogeneous problem

Now look at the problem

$$\begin{cases} a_n x^{(n)} + \cdots + a_0 x = u, \\ x(0) = x'(0) = \cdots = x^{(n-1)}(0) = 0, \end{cases} \tag{9}$$

where $u$ is continuous and of exponential order. Moreover, let $h$ denote the impulse response and $H(s) = 1/(a_n s^n + \cdots + a_0)$ its Laplace transform. We then have

$$(a_n s^n + \cdots + a_0) X(s) = U(s)$$

or

$$X(s) = H(s) U(s).$$

Since multiplication of Laplace transforms corresponds to convolution, this gives the integral formula

$$x(t) = (h * u)(t) = \int_0^t h(s) u(t-s)\, ds \tag{10}$$

for the solution to (9). This result is very important. It holds for any continuous $u$ (even when the Laplace transform of $u$ does not exist), as one can verify directly.

*Exercise* 16. Solve the equation

$$x'' - 3x' + 2x = e^t, \qquad x(0) = x'(0) = 0,$$

using Laplace transform. Verify that the same answer is obtained from (10).

*Exercise* 17. Verify directly that (10) solves (9).

# Answers

(2) $1/s$.

(3a) $1/(s^2-1)$. (3b) $18(s^2-3)/(s^2+9)^3$. (3c) $(s^2+2)/s(s^2+4)$.

(4) $F(s/a)/a$.

(5) $1-\cos t$.

(6a) $(e^{-3t}-e^{-7t})/4\cdot$ (6b) $e^{-t}(\cos 2t-\sin 2t)$. (6c) $\frac{1}{2}\,t\sin t$. (6d)$\frac{1}{2}\,(\sin t - t\cos t)$.

(7a) $\frac{1}{6}\,e^t+\frac{5}{2}\,e^{-t}-\frac{5}{3}\,e^{-2t}$. (7b) $te^{-4t}-\frac{1}{2}\,\cos 4t$.

(7c) $e^t-\frac{1}{2}\,e^{-2t}+\frac{1}{2}\,e^{-t}(\cos t-3\sin t)$. (7d) $-\frac{1}{6}+\frac{1}{2}\,e^t-\frac{1}{2}\,e^{2t}+\frac{1}{6}\,e^{3t}$.

(8) $x(t)=\frac{3}{2}\,e^{t/3}-\frac{1}{2}\,e^t$, $y(t)=-1+\frac{3}{2}\,e^{t/3}+\frac{1}{2}\,e^t$.

(10) $\begin{cases} t, & 0<t\le 1, \\ 2-t, & 1<t<2, \\ 0, & \text{else.} \end{cases}$

(11) $x(t)=f(t)-2H(t-1)f(t-1)+H(t-2)f(t-2)$, where $f(t)=(t+2)e^{-t}+t-2$.

(12) $x(t)=\dfrac{x^3}{12}-\dfrac{x^2}{16}-\dfrac{1}{24}\left(x^4-\left(x-\frac{1}{2}\right)^4 H\left(x-\frac{1}{2}\right)\right)$.

(13) $(1+s-e^s)/s^2(1-e^s)$.

(16) $x(t)=e^{2t}-(t+1)e^t$.