Exercise / Fourier Transform / Solution

E] 60. (a)' Use Fourier transform to compute

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2(x^2+b^2)^2}\,dx. \quad \left(\text{Write } \boxed{F_a(x) = \frac{1}{(x^2+a^2)^2}} \right)$$

Solution. (We compute the Fourier transform of

$F_a(x) = (x^2+a^2)^{-2}$ and use the Plancherel formula).

Let $\boxed{f_a(x) = (x^2+a^2)^{-1}}$. Then [Table 2, 10]:

$$\mathcal{F}_1 : \quad f_a(x) \longrightarrow \frac{\pi}{a} e^{-a|\xi|}. \qquad (1)$$

We perform differentiation on f_a:

$$\mathcal{F}_1 : \quad f_a'(x) = \frac{-2x}{(x^2+a^2)^2} \longrightarrow i\xi \cdot \frac{\pi}{a} e^{-a|\xi|}.$$

To find the F-T of $(x^2+a^2)^{-2}$ we use [Table 2, 6]:

$$\mathcal{F}_1 : \quad -\frac{2x \cdot x}{(x^2+a^2)^2} \longrightarrow \underbrace{i^2 \frac{\pi}{a}\left(e^{-a|\xi|} + \xi(-a)\,\mathrm{sgn}\,\xi\, e^{-a|\xi|}\right)}_{g_a(\xi)}. \quad (2)$$

[Here $\mathrm{sgn}\,\xi = \pm 1$ according to $\xi \gtrless 0$]

However $x^2 = (x^2+a^2) - a^2$ and the above formulas (1) and (2) combined imply

$$g: -\frac{2((x^2+a^2)-a^2)}{(x^2+a^2)^2} = -2\frac{1}{x^2+a^2} \boxed{+2\frac{a^2}{(x^2+a^2)^2}}$$

$$\longrightarrow g_a(\xi),$$

$$\boxed{\frac{2a^2}{(x^2+a^2)^2}} \longrightarrow g_a(\xi) + 2\frac{\pi}{a}e^{-a|\xi|},$$

$$F_a(x) := \frac{1}{(x^2+a^2)^2} \longrightarrow \underbrace{\left(-\frac{1}{2a^2}\right)\left(g_a(\xi) + 2\frac{\pi}{a}e^{-a|\xi|}\right)}_{\hat{F}_a(\xi)}.$$

where

$$\hat{F}_a(\xi) = -\frac{1}{2a^2}e^{-a|\xi|}\left(\div\frac{\pi}{a}(1-a\xi\,\mathrm{sgn}\,\xi) + 2\frac{\pi}{a}\right)$$

$$= -\frac{\pi}{2a^3}e^{-a|\xi|}\left(\div(1-a\xi\,\mathrm{sgn}\,\xi)+2\right).$$

$$= -\frac{\pi}{2a^3}e^{-a|\xi|}\left(1 + a\underbrace{\xi\,\mathrm{sgn}\,\xi}_{|\xi|}\right).$$

Now the integral is, using

$$\langle F_a, F_b\rangle = \langle \hat{F}_a, \hat{F}_b\rangle = \int_{-\infty}^{\infty} \hat{F}_a(\xi)\hat{F}_b(\xi)\,d\xi\,\frac{1}{2\pi}$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{2a^3}\right)\left(\frac{\pi}{2b^3}\right)\underbrace{\int_{-\infty}^{\infty} e^{-(a+b)|\xi|}(1+a|\xi|)(1+b|\xi|)\,d\xi}_{I},$$

and the integral I is

$$I = 2\int_0^\infty e^{-(a+b)\xi}\left(1 + \xi(a+b) + ab\,\xi^2\right)d\xi .$$

Now

$$\int_0^\infty e^{-(a+b)\xi}\,d\xi = \frac{1}{a+b},$$

$$\int_0^\infty e^{-(a+b)\xi}\,\xi\,d\xi = \frac{1}{(a+b)^2}\int_0^\infty e^{-t}\,t\,dt = \frac{1}{(a+b)^2}$$

$$\int_0^\infty e^{-(a+b)\xi}\,\xi^2\,d\xi = \frac{1}{(a+b)^3}\int_0^\infty e^{-t}\,t^2\,dt = \frac{2}{(a+b)^3}.$$

Finally

$$\langle F_a, F_b\rangle = \frac{1}{\cancel{2}\cancel{\pi}}\left(\frac{\cancel{\pi}}{2a^3}\right)\left(\frac{\pi}{2b^3}\right)\cdot\cancel{2}\left(\frac{1}{a+b} + \frac{1}{a+b} + ab\,\frac{2}{(a+b)^3}\right)$$

$$= \frac{\pi}{4a^3b^3}\left(\frac{2}{a+b} + \frac{2ab}{(a+b)^3}\right).$$

[When $a=1$, $b=2$, this gives $\frac{\pi\cdot 11}{3^3\cdot 2^4} = \frac{\pi\cdot 11}{432}$]