[E] 10. Show that

$$\frac{\sin x}{x} = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos(nx), \quad 0 < x < \pi$$

where
$$b_n = \frac{1}{\pi} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin x}{x}\, dx.$$

Use this result to compute $\int_0^\infty \frac{\sin x}{x}\, dx$

Solution. We compute the cos-series of the (even) function $\frac{\sin x}{x}$ on $[0, \pi]$. The coefficients b_n are given by

$$b_n = \frac{2}{\pi} \int_0^\pi \frac{\sin x}{x} \cos(nx)\, dx = \frac{2}{\pi} I, \quad \text{the integral } I \text{ is}$$

$$I = \frac{1}{2} \int_0^\pi \frac{1}{x} \left(\sin(1+n)x + \sin(1-n)x \right) dx$$

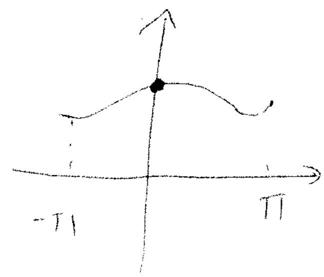
$$= \frac{1}{2} \int_0^\pi \frac{1}{x} \left(\sin \underbrace{(n+1)x}_{y} - \sin \underbrace{(n-1)x}_{y} \right) dx$$

$$= \frac{1}{2} \left\{ \int_0^{(n+1)\pi} \frac{\sin y}{y}\, dy - \int_0^{(n-1)\pi} \frac{\sin \eta}{\eta}\, dy \right\} \quad \left[\begin{array}{l} \text{Change variables} \\ y = (n+1)x \\ \eta = (n-1)x \end{array} \right]$$

$$= \frac{1}{2} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin y}{\eta}\, dy.$$

Thus $$b_n = \frac{1}{\pi} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin y}{\eta}\, dy.$$

We evaluate the series at $x = 0$



$$\lim_{x \to \pm 0} \frac{\sin x}{x} = 1$$

$$\Rightarrow \quad 1 = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n . \quad (*)$$

$x=0$, continuous pt

The RHS is

$$\frac{b_0}{2} + \sum_{n=1}^{\infty} b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin y}{y}\, dy + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin y}{y}\, dy$$

$$= \frac{1}{\pi} \left(\int_0^{\pi} \frac{\sin y}{y}\, dy + \sum_{n=1}^{\infty} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin y}{y}\, dy \right)$$

Now the intervals of the integrations are

Thus the sum of the integration is twice of $\int_0^{\infty}$, and the identity $(*)$ is

$$1 = \frac{2}{\pi} \int_0^{\infty} \frac{\sin y}{y}\, dy, \qquad \boxed{\int_0^{\infty} \frac{\sin y}{y}\, dy = \frac{\pi}{2}}$$

(See also Table 2 Entry 13 p. 223)