# Additional Elementary Exercises on Fourier Transform (Chapter 7)

1. Let $p \in \mathbb{R}$ and consider the function $f(x) = |x|^p$, $x \in \mathbb{R}$.

(a) Prove that the function $f$ is never in $L^1(\mathbb{R})$ for any $p$

(b) Consider now instead $f$ as a function on the interval $(0, \infty)$ (resp. $(0, 1)$). Find conditions on $p$ so that $f \in L^1(0, \infty)$ (resp. $L^1(0, 1)$).

2. Prove the formula for Fourier transform of $f(x) = \sin x/x$ (see [F] Table 2. Entry 13) by completing the following steps.

(a). Solve the problem [E] 10, and prove that $\int_{-\infty}^{\infty} f(x)dx = 2\int_0^\infty f(x)dx = \pi$. (This was done in an exercise session. Check your or your fellow students' notes)

(b) Prove $\int_{-\infty}^{\infty} \frac{\sin(bx)}{x} dx = \operatorname{sgn}(b)\pi$. Here $\operatorname{sgn}(b)$ is the sign function, namely it's $\pm$ according to $b > 0$ or $b < 0$. (Hint. This follows immediately from (a) by a change of variables.)

(c) Compute the Fourier transform for $f(x)$. (Hint. $f(x)$ is an even function so

$$\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx = \int_{-\infty}^{\infty} f(x)\cos(-\xi x)dx = \int_{-\infty}^{\infty} f(x)\cos(\xi x)dx.$$

Now use $\sin x \cos(\xi x) = \frac{1}{2}(\sin(\xi+1)x - \sin(\xi-1)x)$ and (b)).

3. Denote $\chi_{[a,b]}(x)$ the characteristic function for the interval $[a, b]$, namely it is 1 on the interval and zero elsewhere. Consider the following functions

$$f(x) = \sum_{n=1}^{\infty} \chi_{[n, n+\frac{1}{n^2}]}(x),$$

$$g(x) = \sum_{n=1}^{\infty} \chi_{[n, n+\frac{1}{n}]}(x).$$

(a) Prove that $f \in L^1(\mathbb{R})$, and $g \in L^2(\mathbb{R})$, but $g \notin L^1(\mathbb{R})$.

(b). Prove that $\hat{f}(\xi)$ is a continuous function, and that $\hat{g}(\xi)$ is not continuous (actually it's not well-defined as point-wise function in the usual sense.)