

**Solution (with details). Exam in MMG710/TMA362 Fourier Analysis,  
2014-10-27**

1. In which space  $C^k$  are the following periodic functions? Find the best (i.e. the largest)  $k$ .

$$(a) \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^2}, \quad (b) \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{5^n}, \quad (c) \sum_{n=1}^{\infty} \frac{\sin(2^n\theta)}{5^n}.$$

Motivate your answers.

Solution We use the Theorem on term-wise differentiation of Fourier series. Write the given series as  $f(\theta)$ . In the cases (a)-(b)  $f(\theta)$  is a well-defined convergent series and thus a well-defined function.

(a) Differentiating the series formally term-wise we get a series

$$\sum_{n=1}^{\infty} \frac{d}{d\theta} \frac{\cos(n\theta)}{n^2} = - \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n}$$

which is convergent but not absolutely convergent. Thus (the best we can say)  $f$  is  $C$  (but not  $C^1$ ).

(b). The series  $\sum_{n=-\infty}^0 \frac{(in)^j e^{in\theta}}{5^n}$  is divergent. Thus the series has no regularity. (However the series  $\sum_{n=1}^{\infty} \frac{(in)^j e^{in\theta}}{5^n}$  is  $C^\infty$ , since differentiating term-wise of the series  $j$ -times results in an absolutely convergent series, for  $\sum_{n=1}^{\infty} \frac{n^j}{5^n} < \infty$ .)

(c) We perform differentiation twice on the series, and find

$$- \sum_{n=1}^{\infty} \frac{(2^n)^2 \sin(2^n\theta)}{5^n},$$

which is absolutely dominated by  $\sum_{n=1}^{\infty} \frac{(2^n)^2}{5^n} = \sum_{n=1}^{\infty} (\frac{4}{5})^n < \infty$ , whereas differentiating one more time it is

$$- \sum_{n=1}^{\infty} \frac{2^{3n} \sin(2^n\theta)}{5^n},$$

which is divergent, e.g. for  $\theta = \frac{\pi}{3}, \frac{\pi}{5}, \frac{\pi}{7}$  etc. Thus  $f \in C^2$ .

2. Compute the following integral

$$\int_{-\infty}^{\infty} \frac{\sin(x) \cos(2x)}{x(x^2 + 1)} dx$$

Solution Write  $\sin(x) \cos(2x) = \frac{1}{2}(\sin(3x) - \sin(x))$  and thus

$$I := \int_{-\infty}^{\infty} \frac{\sin(x) \cos(2x)}{x(x^2 + 1)} dx = \frac{1}{2} \left( \int_{-\infty}^{\infty} \frac{\sin(3x)}{x(x^2 + 1)} dx - \int_{-\infty}^{\infty} \frac{\sin(x)}{x(x^2 + 1)} dx \right)$$

We compute, for any  $a > 0$ , the integral  $\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+1)} dx$ . Write  $f(x) = \frac{\sin(ax)}{x}$ ,  $g(x) = (x^2 + 1)^{-1}$  and use Plancherel formula:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2 + 1)} dx &= (f, g) = \frac{1}{2\pi} (\hat{f}, \hat{g}) \\ &= \frac{1}{2\pi} (\pi \chi_a, \pi e^{-|\cdot|}) = \frac{\pi}{2} \int_{-a}^a e^{-|\xi|} d\xi \\ &= \pi \int_0^a e^{-\xi} d\xi = \pi(1 - e^{-a}) \end{aligned}$$

Thus taking  $a = 3, 1$  we find

$$I = \frac{1}{2} (\pi(1 - e^{-3}) - \pi(1 - e^{-1})) = \frac{\pi}{2} (e^{-1} - e^{-3}).$$

3. Solve the following ordinary differential equation

$$u''(t) - 4u(t) = f(t), \quad u(0) = 0, \quad u'(0) = 1,$$

where

$$f(t) = H(t - 1) = \begin{cases} 1, & t \geq 1 \\ 0, & \text{else} \end{cases}$$

Solution We apply the Laplace transform  $\mathcal{L}$  to the equation, writing  $\mathcal{L}u(z) = U(z)$ ,

$$z^2U(z) - u'(0) - zu(0) - 4U(z) = 8\mathcal{L}[H(t - 1)](z) = 8\frac{e^{-z}}{z}.$$

$$(z^2 - 4)U(z) - 1 = 8\frac{e^{-z}}{z}.$$

Solve  $U(z)$  and perform partial fractional decompositions:

$$U(z) = 8e^{-z} \frac{1}{z(z-2)(z+2)} + \frac{1}{z^2 - 2^2} = e^{-z} \left( \frac{-2}{z} + \frac{1}{z-2} + \frac{1}{z+2} \right) + \frac{1}{z^2 - 2^2}$$

Its inverse transform gives the solution

$$u(t) = H(t-1)(-2 + e^{2(t-1)} + e^{-2(t-1)}) + \frac{1}{2} \sinh(2t) = 2H(t-1)(-1 + \cosh 2(t-1)) + \frac{1}{2} \sinh(2t)$$

4. Solve the following inhomogeneous wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} + t \sin(2x), & t > 0, \quad 0 < x < \pi \\ u(0, t) = 0, \quad u(\pi, t) = 0, & t > 0 \\ u(x, 0) = x(\pi - x), & 0 < x < \pi \end{cases}$$

You may use (without proof) the following Fourier sine series on  $(-\pi, \pi)$

$$\theta(\pi - |\theta|) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\theta}{(2n-1)^3}.$$

Solution . We find first a special solution solving the inhomogeneous equation and preserving the homogeneous condition. Ansatz

$$\boxed{w(x, t) = \frac{1}{c^2} t \sin(2x)}.$$

Then  $w_{tt} = 0$  and  $c^2 w_{xx} = -t \sin(2x)$ , namely  $c^2 w_{xx} + t \sin(2x) = 0$ ,  $w$  solves indeed the inhomogeneous equation along with the boundary condition since  $\sin(2x) = 0$  for  $x = 0, \pi$ . Now writing  $u = w + v$  and  $u_t(x, 0) = g(x)$  the function  $v$  satisfies

$$\begin{cases} v_{tt} = c^2 v_{xx} \\ v(0, t) = 0, \quad v(\pi, t) = 0 \\ v(x, 0) = x(\pi - x), \quad v_t(x, 0) = g(x) - \frac{2}{c^2} \sin(2x), \end{cases}$$

Let  $\beta_n$  be the Fourier sine coefficients of  $g$  and write

$$\boxed{\alpha_n = \begin{cases} \beta_n, & n \neq 2 \\ \beta_2 - \frac{2}{c^2}, & n = 2. \end{cases}}$$

The Fourier sine series of  $v(x, 0) = x(\pi - x)$  on  $(0, \pi)$  is given by  $\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}$ , whereas the Fourier sine series of  $v_t(x, 0)$  is  $\sum_{n=1}^{\infty} \alpha_n \sin nx$ . The solution for  $v$  is given by

$$v(x, t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)ct) \sin((2n-1)x)}{(2n-1)^3} + \sum_{n=1}^{\infty} \frac{1}{nc} \alpha_n \sin(nct) \sin nx$$

Answer:  $u(x, t) = w(x, t) + v(x, t)$ .

5. Evaluate the sum of the following series by using the above Fourier expansion. Motivate your answer.

$$(a) \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}, \quad (b) \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}.$$

Solution The function  $f(x) = x(\pi - |x|)$  on  $(-\pi, \pi)$  has continuous derivative and piece-wise second order derivative:  $f'(x) = \pi - 2x \operatorname{sgn}(x)$  is continuous,  $f''(x) = 2 \operatorname{sgn}(x)$  is piece-wise continuous. Thus  $f'(x)$  has its Fourier series given by term-wise differentiation and the series converges to  $f'(x)$ ,

$$f'(x) = \pi - 2x \operatorname{sgn}(x) = \theta(\pi - |\theta|) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}.$$

Taking  $x = 0$  we find

$$\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \pi, \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \pi^2/8,$$

We apply Parseval's (Pythagoras') theorem to the Fourier expansion of  $f(x)$  on  $(-\pi, \pi)$

$$\int_{-\pi}^{\pi} f(x)^2 dx = \pi \left(\frac{8}{\pi}\right)^2 S, \quad S := \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6},$$

since  $\int_{-\pi}^{\pi} \sin^2(2n-1)x dx = \pi$ . Now the left hand side is

$$\int_{-\pi}^{\pi} f(x)^2 dx = 2 \int_{-0}^{\pi} x^2(\pi-x)^2 dx = 2 \int_{-0}^{\pi} (\pi^2 x^2 - 2\pi x^3 + x^4) dx = 2\left(\pi^2 \frac{\pi^3}{3} - 2\pi \frac{\pi^4}{4} + \frac{\pi^5}{5}\right) = \frac{\pi^5}{15}.$$

Thus

$$S = \frac{\pi^6}{2^6 \cdot 15}.$$

6. Formulate and prove *the Theorem on Uniform Convergence for Fourier Series* of  $2\pi$ -periodic  $C^1$ -functions.

Solution See the textbook.