Sturm – Liouville Problem

Ex. ([E] 52') Find the eigenvalues and eigenfunctions for the following S-L-prob.

$$\begin{cases} -e^{-4x}\dfrac{d}{dx}\left(e^{4x}\dfrac{du}{dx}\right) = \lambda u, & 0 < x < 1 \quad \text{(PDE)} \\ u(0) = 0, \quad u'(1) = 0 & \text{(BV)} \end{cases}$$

Solution (As only the exponential functions are involved in the PDE, we try to solve the eq. using also the exponential functions).

Ansatz $u(x) = e^{\mu x}$.

(PDE) $$-e^{-4x}\frac{d}{dx}\left(e^{(4+\mu)x}\mu\right) = \lambda e^{\mu x}$$

$$\boxed{-\mu(\mu+4) = \lambda} \qquad \mu = \mu_\pm = -2 \pm \sqrt{4-\lambda}$$

The general solution is (assuming first $\lambda \neq 4$)

$$u(x) = A e^{\mu_+ x} + B e^{\mu_- x}$$

(BV) $0 = u(0) = A + B, \quad B = -A$

$$0 = u'(1) = A(\mu_+ e^{\mu_+} - \mu_- e^{\mu_-})$$

$$\mu_+ e^{\mu_+} = \mu_- e^{\mu_-} \underset{(\mu_- \neq 0)}{\Longleftrightarrow} \frac{\mu_+}{\mu_-} = e^{-2\sqrt{4-\lambda}}$$

Write $\boxed{\sqrt{4-\lambda} = 2\sigma}$

The equation $\frac{\mu_+}{\mu_-} = e^{-2\sqrt{4-\lambda}}$ is now

$$\frac{1-\sigma}{1+\sigma} = e^{-4\sigma}$$

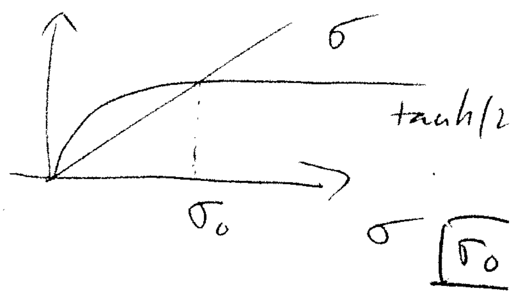
$$\Leftrightarrow \quad \frac{1-e^{-4\sigma}}{1+e^{-4\sigma}} = \sigma \quad \Leftrightarrow \quad \frac{e^{2\sigma}-e^{-2\sigma}}{e^{2\sigma}+e^{2\sigma}} = \sigma$$

$$\Leftrightarrow \quad \boxed{\tanh(2\sigma) = \sigma} \qquad (*)$$

Case 1 $\quad 4-\lambda>0$, σ is a real number, $\sigma>0$

$(*)$: $\boxed{\sigma = \tanh(2\sigma)}$

which has only one solution, denoted by σ_0

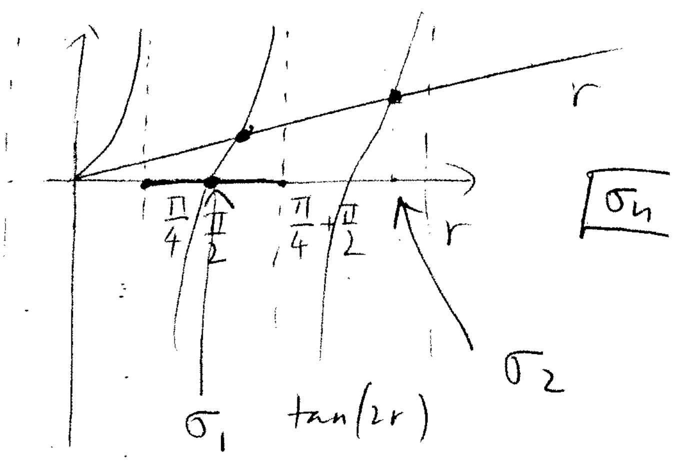


Case 2 $\quad 4-\lambda<0$, $\sigma = \frac{1}{2}\sqrt{4-\lambda} = ir$, $r>0$ real.

$(*)$ $\boxed{\tan(2r) = r}$

$\exists$ infinitely many solutions $\{r_n\}_{n=1}^{\infty}$

$$r_n \in \left(\frac{n\pi}{2}, \frac{\pi}{4}+\frac{n\pi}{2}\right)$$



Case 3 $\quad \lambda=4$. Now the solution u is instead

$$u(x) = Ae^{-2x} + Bxe^{-2x}$$

(BV): $\quad A=0 \quad B=0 \quad$ Not an eigenvalue.

Answer $\quad \lambda = 4(1-\sigma_0^2)$, $u(x) = e^{-2x}\sinh(2\sigma_0 x)$

$\lambda = 4(1+\sigma_n^2)$, $u(x) = e^{-2x}\sin(2\sigma_n x)$.