Fourier analysis (MMG710/TMA362)

Time: 2013-10-26, 8:30–12:30.
Tools: Only the attached sheet of formulas. No calculator or handbook is allowed.
Questions: Jakob Hultgren, 0703-088304.
Grades: Each problem gives 4 points. For MMG710 grades are G (12-17 points) and VG (18-24). For TMA362 grades are 3 (12-14 points), 4 (15-17) and 5 (18-24).

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1 Find all solutions of the form $u(x,t) = X(x)T(t)$ to the equation

$$u_{xx} + u_{xt} = 0.$$

2 Use Laplace transform to solve the initial value problem

$$x''(t) - 2x'(t) + x(t) = 1, \qquad x'(0) = 2, \qquad x(0) = 0.$$

3 If $\hat{f}(\xi) = \dfrac{\xi}{|\xi|^3 + 1}$, show that $\overline{f(x)} = -f(x)$ for $x \in \mathbb{R}$.

Use this to compute $\displaystyle\int_{-\infty}^{\infty} f(x)^2\, dx$.

4 Expand $\sin(x/2)$ as a Fourier cosine series on the interval $0 < x < \pi$. Use the result to compute the series

$$\left(1 - \frac{1}{3}\right)^2 + \left(\frac{1}{3} - \frac{1}{5}\right)^2 + \left(\frac{1}{5} - \frac{1}{7}\right)^2 + \cdots.$$

It may be helpful to note that $\dfrac{1}{2n-1} - \dfrac{1}{2n+1} = \dfrac{2}{4n^2-1}$.

5 (a) Formulate and prove Lagrange's identity for a Sturm–Liouville-operator $L(f) = (rf')' + pf$ (where p and r are real-valued functions). (2p)

(b) Use Lagrange's identity to prove that

$$\int_0^{\pi} \sin(mx)\sin(nx)\, dx = 0$$

for m and n any two distinct positive integers. (2p)

6 Prove that a continuous 2π-periodic function, with continuous derivative, is equal to its Fourier series.

Fourier analysis (MMG710/TMA362)

2013-10-26, Solutions

1 Find all solutions of the form $u(x,t) = X(x)T(t)$ to the equation

$$u_{xx} + u_{xt} = 0.$$

The equation can be written $X''(x)T(t) + X'(x)T'(t) = 0$, which gives

$$\frac{X''(x)}{X'(x)} = -\frac{T'(t)}{T(t)} = \lambda,$$

where λ is a constant. The equation $X'' = \lambda X'$ has solutions $X(x) = Ax + B$ if $\lambda = 0$ and $X(x) = Ae^{\lambda x} + B$ else. The equation $T' = -\lambda T$ has solutions $T(t) = Ce^{-\lambda t}$, where we can choose $C = 1$ without loss of generality. In conclusion, the solutions are $u(x,t) = Ax + B$ and $u(x,t) = (Ae^{\lambda x} + B)e^{-\lambda t}$.

2 Use Laplace transform to solve the initial value problem

$$x''(t) - 2x'(t) + x(t) = 1, \qquad x'(0) = 2, \qquad x(0) = 0.$$

Applying Laplace transform to the equation gives

$$s^2X(s) - 2 - 2sX(s) + X(s) = \frac{1}{s} \qquad \Longleftrightarrow \qquad X(s) = \frac{1+2s}{s(s-1)^2} = \frac{1}{s} - \frac{1}{s-1} + \frac{3}{(s-1)^2}.$$

The inverse Laplace transform gives the answer $x(t) = 1 - e^t + 3te^t$.

3 If $\hat{f}(\xi) = \dfrac{\xi}{|\xi|^3 + 1}$, show that $\overline{f(x)} = -f(x)$ for $x \in \mathbb{R}$. Use this to compute $\displaystyle\int_{-\infty}^{\infty} f(x)^2\,dx$.

By Fourier's inversion formula,

$$\overline{f(x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\xi}{|\xi|^3+1}\,e^{-ix\xi}\,d\xi = (\xi \mapsto -\xi) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\xi}{|\xi|^3+1}\,e^{ix\xi}\,d\xi = -f(x).$$

Multiplying this identity with $f(x)$ gives $|f(x)|^2 = -f(x)^2$. By Parseval's formula,

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\,d\xi = \frac{1}{\pi}\int_0^{\infty} \frac{\xi^2}{(\xi^3+1)^2}\,d\xi = \frac{1}{\pi}\left[-\frac{1}{3(\xi^2+1)}\right]_0^{\infty} = \frac{1}{3\pi},$$

where we used that the integrand is even. Thus, $\int_{-\infty}^{\infty} f(x)^2\,dx = -1/3\pi$.

4 Expand $\sin(x/2)$ as a Fourier cosine series on the interval $0 < x < \pi$. Use the result to compute the series

$$\left(1 - \frac{1}{3}\right)^2 + \left(\frac{1}{3} - \frac{1}{5}\right)^2 + \left(\frac{1}{5} - \frac{1}{7}\right)^2 + \cdots.$$

It may be helpful to note that $\dfrac{1}{2n-1} - \dfrac{1}{2n+1} = \dfrac{2}{4n^2-1}$.

The Fourier cosine series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi \sin\left(\frac{x}{2}\right)\cos(nx)\,dx = \frac{1}{\pi}\int_0^\pi \left(\sin\left(\frac{(2n+1)x}{2}\right) - \sin\left(\frac{(2n-1)x}{2}\right)\right)dx$$

$$= \frac{2}{\pi}\left[-\frac{\cos\frac{(2n+1)x}{2}}{2n+1} + \frac{\cos\frac{(2n-1)x}{2}}{2n-1}\right]_0^\pi = \frac{2}{\pi}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) = -\frac{4}{\pi(4n^2-1)}.$$

Thus, the Fourier cosine series is

$$\sin\frac{x}{2} = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^\infty \frac{\cos(nx)}{4n^2-1}, \qquad 0 < x < \pi.$$

In order to compute the series

$$S = \sum_{n=1}^\infty \frac{4}{(4n^2-1)^2},$$

we apply Parseval's formula

$$\int_0^\pi \sin^2\frac{x}{2}\,dx = \frac{\pi}{4}|a_0|^2 + \frac{\pi}{2}\sum_{n=1}^\infty |a_n|^2 = \frac{4}{\pi} + \frac{2}{\pi}S,$$

where the left-hand side equals

$$\int_0^\pi \frac{1-\cos x}{2}\,dx = \left[\frac{x-\sin x}{2}\right]_0^\pi = \frac{\pi}{2}.$$

Solving for S gives $S = (\pi^2 - 8)/4$.

5 (a) Formulate and prove Lagrange's identity for a Sturm–Liouville-operator $L(f) = (rf')' + pf$ (where p and r are real-valued functions).

(b) Use Lagrange's identity to prove that

$$\int_0^\pi \sin(mx)\sin(nx)\,dx = 0$$

for m and n any two distinct positive integers.

(a) See the course literature.
(b) With $r = 1$ and $p = 0$, Lagrange identity is

$$\int_0^\pi f''(x)\overline{g(x)}\,dx - \int_0^\pi f(x)\overline{g''(x)}\,dx = [f'\bar{g} - f\bar{g}']_0^\pi.$$

Plugging in $f(x) = \sin(mx)$ and $g(x) = \sin(nx)$, this reads

$$-m^2\int_0^\pi \sin(mx)\sin(nx)\,dx + n^2\int_0^\pi \sin(mx)\sin(nx)\,dx$$
$$= [m\cos(mx)\sin(nx) - n\sin(mx)\cos(mx)]_0^\pi,$$

which simplifies to

$$(n^2 - m^2)\int_0^\pi \sin(mx)\sin(nx)\,dx = 0.$$

When $m \neq n$, it follows that the integral is zero.

6 Prove that a continuous 2π-periodic function, with continuous derivative, is equal to its Fourier series.

See Theorem 2.1 in Folland. (Since we assume that f and f' are continuous, the proof can be simplified a bit, but it is of course not necessary to observe that.)