Fourier analysis (MMG710/TMA362)

Time: 2014-01-11, 8:30–12:30.
Tools: Only the attached sheet of formulas. No calculator or handbook is allowed.
Questions: Oskar Hamlet 0703-088304
Grades: Each problem gives 4 points. For MMG710 grades are G (12-17 points) and VG (18-24). For TMA362 grades are 3 (12-14 points), 4 (15-17) and 5 (18-24).

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1 Compute the Fourier transform of the function $f(x) = \dfrac{e^{-|x|}\sin(x)}{x}$.

2 Find numbers A and B so that

$$\int_0^2 \left|x^2 - Ax - B\right|^2 dx$$

is minimal.

3 Compute the Laplace transform of the function

$$f(t) = \begin{cases} t, & 0 < t < 1, \\ 2-t, & 1 < t < 2, \\ 0, & t > 2. \end{cases}$$

Use this to solve the equation

$$x''(t) + x(t) = f(t), \qquad x(0) = 0, \quad x'(0) = 1.$$

4 Solve the inhomogeneous heat equation

$$\begin{aligned} u_t &= 2u_{xx} + \cos x, \\ u_x(0,t) &= u_x(\pi,t) = 0, \\ u(x,0) &= \sin^2 x \end{aligned}$$

for $u = u(x,t)$ in the region $0 < x < \pi$, $t > 0$.

5 Define what it means for a series $\sum_{n=1}^{\infty} f_n(x)$ to converge *uniformly* to $f(x)$ on $\mathbb{R}$. Formulate and prove a statement on uniform convergence of Fourier series.

6 In a table of Fourier series, you find the entry

$$\sinh(x) = \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^2+1}\sin(nx), \qquad 0 < x < \pi.$$

Use this to compute the sum

$$\sum_{n=1}^{\infty}\frac{1}{(n^2+1)^2}.$$

Fourier analysis (MMG710/TMA362)

2013-10-26, Solutions

1 Compute the Fourier transform of the function $f(x) = \dfrac{e^{-|x|}\sin(x)}{x}$.

Using the table, we find that

$$\mathcal{F}\left(e^{-|x|}\frac{\sin(x)}{x}\right) = \frac{1}{2\pi}\mathcal{F}\left(e^{-|x|}\right) * \mathcal{F}\left(\frac{\sin(x)}{x}\right) = \chi(\xi) * \frac{1}{\xi^2+1} = \int_{-\infty}^{\infty} \frac{\chi(\xi - t)}{t^2+1}\,dt$$
$$= \int_{\xi-1}^{\xi+1} \frac{1}{t^2+1}\,dt = \arctan(\xi+1) - \arctan(\xi-1),$$

where $\chi(t) = 1$ for $|t| < 1$ and 0 else.

2 Find numbers A and B so that

$$\int_0^2 \left|x^2 - Ax - B\right|^2 dx$$

is minimal.

We need to approximate x^2 by a first degree polynomial in the norm of $L^2([0,2])$. We first find an orthogonal basis for the space of first degree polynomials. We take $e_1 = 1$. With $e_2 = Cx + D$, we have

$$\langle e_1, e_2\rangle = \int_0^2 (Cx + D)\,dx = 2C + 2D.$$

Thus, choosing $e_2 = x - 1$, we have $\langle e_1, e_2\rangle = 0$. We can now write the approximating polynomial q as

$$q = \frac{\langle x^2, e_1\rangle}{\langle e_1, e_1\rangle}\,e_1 + \frac{\langle x^2, e_2\rangle}{\langle e_2, e_2\rangle}\,e_2,$$

where

$$\begin{aligned}
\langle x^2, e_1\rangle &= \int_0^2 x^2\,dx = \frac{8}{3},\\
\langle e_1, e_1\rangle &= \int_0^2 dx = 2,\\
\langle x^2, e_2\rangle &= \int_0^2 (x^3 - x^2)\,dx = \frac{4}{3},\\
\langle e_1, e_1\rangle &= \int_0^2 (x-1)^2\,dx = \frac{2}{3}.
\end{aligned}$$

This gives

$$q = \frac{4}{3} + 2(x-1) = 2\left(x - \frac{1}{3}\right),$$

so $A = 2$, $B = -2/3$.

3 Compute the Laplace transform of the function

$$f(t) = \begin{cases} t, & 0 < t < 1,\\ 2 - t, & 1 < t < 2,\\ 0, & t > 2. \end{cases}$$

Use this to solve the equation

$$x''(t) + x(t) = f(t), \qquad x(0) = 0, \quad x'(0) = 1.$$

We first write f in terms of Heaviside's function as

$$f(t) = t - 2(t-1)H(t-1) + (t-2)H(t-2),$$

which gives the Laplace transform

$$F(s) = \frac{1 - 2e^{-s} + e^{-2s}}{s}.$$

The Laplace transform of the differential equation is then

$$(s^2+1)X(s) - 1 = \frac{1 - 2e^{-s} + e^{-2s}}{s^2},$$

which after a partial fraction decomposition can be written

$$X(s) = \frac{1}{s^2+1} + \left(1 - 2e^{-s} + e^{-2s}\right)\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right).$$

We can then read off the inverse Laplace transform

$$x(t) = t - 2\left(t - 1 - \sin(t-1)\right)H(t-1) + \left(t - 2 - \sin(t-2)\right)H(t-2).$$

4 Solve the inhomogeneous heat equation

$$\begin{aligned} u_t &= 2u_{xx} + \cos x, \\ u_x(0,t) &= u_x(\pi,t) = 0, \\ u(x,0) &= \sin^2 x \end{aligned}$$

for $u = u(x,t)$ in the region $0 < x < \pi$, $t > 0$.

We first look for a stationary solution $u_0 = u_0(x)$ to the equation with boundary conditions. The equation gives

$$0 = 2u_0''(x) + \cos x,$$

with solutions $u_0(x) = \frac{1}{2}\cos x + Ax + B$. The boundary conditions $u_0'(0) = u_0'(\pi) = 0$ gives $A = 0$, whereas B is arbitrary. We choose $B = 0$, $u_0(x) = \frac{1}{2}\cos x$. Writing $u = u_0 + v$, we find that v satisfies the homogeneous equation

$$\begin{aligned} v_t &= 2v_{xx}, \\ v_x(0,t) &= v_x(\pi,t) = 0, \\ v(x,0) &= \sin^2 x - \frac{1}{2}\cos x. \end{aligned}$$

To solve this we should expand $v(x,0)$ as a Fourier cosine series. But since

$$\sin^2 x - \frac{1}{2}\cos x = \frac{1}{2} - \cos x,$$

that series is in fact a finite sum. The solution is $v(x,t) = \frac{1}{2} - e^{-2t}\cos x$ and the answer to the original problem $u(x,t) = \frac{1}{2} - e^{-2t}\cos x + \frac{1}{2}\cos x$.

5 Define what it means for a series $\sum_{n=1}^{\infty} f_n(x)$ to converge *uniformly* to $f(x)$ on $\mathbb{R}$. Formulate and prove a statement on uniform convergence of Fourier series.

See Folland, Thm. 2.5.

6 In a table of Fourier series, you find the entry

$$\sinh(x) = \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^2+1}\sin(nx), \qquad 0 < x < \pi.$$

Use this to compute the sum

$$\sum_{n=1}^{\infty}\frac{1}{(n^2+1)^2}.$$

We will first integrate the series and then apply Parseval's formula. By Folland, Thm. 2.4, we may integrate termwise to obtain the Fourier cosine series

$$\cosh(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx),$$

where we can read off

$$a_n = \frac{2\sinh\pi}{\pi}\frac{(-1)^n}{n^2+1}, \qquad n \geq 1$$

from the given series. The constant term is obtained from

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\cosh(x)\,dx = \frac{2\sinh(\pi)}{\pi}.$$

The relevant version of Parseval's formula is

$$\int_0^{\pi}\cosh^2 x\,dx = \frac{\pi}{4}|a_0|^2 + \frac{\pi}{2}\sum_{n=1}^{\infty}|a_n|^2,$$

where the left-hand side is

$$\int_0^{\pi}\frac{1+\cosh(2x)}{2}\,dx = \left[\frac{x}{2} + \frac{\sinh(2x)}{4}\right]_0^{\pi} = \frac{\pi}{2} + \frac{\sinh(2\pi)}{4}.$$

Plugging in the explicit expressions for a_n, we obtain after simplification

$$\sum_{n=1}^{\infty}\frac{1}{(n^2+1)^2} = \frac{\pi(2\pi+\sinh(2\pi))}{8\sinh^2\pi} - \frac{1}{2}.$$