Fourier analysis (MMG710/TMA362)

Time: 2014-08-18, 8:30–12:30.

Tools: Only the attached sheet of formulas. No calculator or handbook is allowed.

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Grades: Each problem gives 4 points. For MMG710 grades are G (12-17 points) and VG (18-24). For TMA362 grades are 3 (12-14 points), 4 (15-17) and 5 (18-24).

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1 Find a function f such that, for $s > 0$,

$$\int_0^\infty f(t)e^{-ts}\,dt = \frac{e^{-s}}{s^2(s+1)}.$$

2 Let $f(x) = 1$ for $0 < x < \pi/2$ and $f(x) = 0$ for all other values of x. Find (as a Fourier series) the solution $u(x,t)$, $0 < x < \pi$, $t > 0$, to the problem

$$\begin{cases} u_t = 3u_{xx}, \\ u_x(0,t) = u_x(\pi,t) = 0, \\ u(x,0) = f(x). \end{cases}$$

3 Using Fourier transform, compute the integral

$$\int_{-\infty}^\infty \frac{\cos(x)}{x^2 - 2x + 2}\,dx.$$

4 Suppose that f is given by the Fourier cosine series

$$f(x) = \sum_{n=0}^\infty \frac{\cos(nx)}{2^n}.$$

Compute the integral

$$\int_0^\pi f(x)^2\,dx.$$

5 Formulate and prove Bessel's inequality for Fourier series.

6 Consider the Sturm-Liouville problem

$$u'' = \lambda u, \qquad u(0) = -u(\pi), \quad u'(0) = -u'(\pi).$$

(a) Prove that the problem is *symmetric* (or *self-adjoint* in the terminology of Folland), in the sense that

$$\langle u'', v\rangle = \langle u, v''\rangle,$$

for sufficiently differentiable functions u and v satisfying the boundary conditions. (1p)

(b) Find all solutions to the problem. (3p)

Fourier analysis (MMG710/TMA362)

2014-08-18, Solutions

1 Find a function f such that, for $s > 0$,

$$\int_0^\infty f(t)e^{-ts}\,dt = \frac{e^{-s}}{s^2(s+1)}.$$

We need to find the inverse Laplace transform of

$$F(s) = \frac{e^{-s}}{s^2(s+1)}.$$

Ignoring for a moment the exponential factor, we have the partial fraction expansion

$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1},$$

with inverse Laplace transform $t - 1 + e^{-t}$. By the shift rule, the desired function is

$$f(t) = H(t-1)\big((t-1) - 1 + e^{-(t-1)}\big) = H(t-1)\big(t - 2 + e^{1-t}\big).$$

2 Let $f(x) = 1$ for $0 < x < \pi/2$ and $f(x) = 0$ for all other values of x. Find (as a Fourier series) the solution $u(x,t)$, $0 < x < \pi$, $t > 0$, to the problem

$$\begin{cases} u_t = 3u_{xx}, \\ u_x(0,t) = u_x(\pi,t) = 0, \\ u(x,0) = f(x). \end{cases}$$

In view of the boundary conditions, we should expand f as a Fourier cosine series

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^\infty A_n \cos(nx),$$

where

$$A_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi/2} \cos(nx)\,dx.$$

This gives $A_0 = 1$ and, for $n \geq 1$,

$$A_n = \frac{2\sin(n\pi/2)}{\pi n} = \begin{cases} 0, & n \text{ even}, \\ 2(-1)^k/\pi(2k+1), & n = 2k+1. \end{cases}$$

In conclusion,

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^\infty \frac{(-1)^k \cos((2k+1)x)}{2k+1}.$$

The solution to the given problem is then

$$u(x,t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^\infty \frac{(-1)^k \cos((2k+1)x)e^{-3(2k+1)^2 t}}{2k+1}.$$

3 Using Fourier transform, compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 - 2x + 2}\, dx.$$

Let

$$f(x) = \frac{1}{x^2 - 2x + 2} = \frac{1}{(x-1)^2 + 1}.$$

By standard rules for the Fourier transform,

$$\hat{f}(\xi) = \pi e^{-i\xi - |\xi|}.$$

We now observe that

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 - 2x + 2}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ix} + e^{-ix}}{x^2 - 2x + 2}\, dx = \frac{\hat{f}(-1) + \hat{f}(1)}{2} = \frac{\pi(e^{i-1} + e^{-i-1})}{2}$$
$$= \frac{\pi \cos(1)}{e}.$$

4 Suppose that f is given by the Fourier cosine series

$$f(x) = \sum_{n=0}^{\infty} \frac{\cos(nx)}{2^n}.$$

Compute the integral

$$\int_0^{\pi} f(x)^2\, dx.$$

Parseval's formula for cosine series is

$$\int_0^{\pi} |f(x)|^2 = \frac{\pi}{4}|A_0|^2 + \frac{\pi}{2}\sum_{n=1}^{\infty} |A_n|^2,$$

where

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx).$$

In the case at hand, f is real-valued, so $|f(x)|^2 = f(x)^2$. Moreover, $A_0 = 2$ and $A_n = 2^{-n}$ for $n \geq 1$. We find that the integral is equal to

$$\pi + \frac{\pi}{2}\sum_{n=1}^{\infty} 4^{-n}.$$

The series is geometric and given by

$$\sum_{n=1}^{\infty} 4^{-n} = \frac{1}{4} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{1}{3}.$$

We conclude that the integral is equal to $7\pi/6$.

5 Formulate and prove Bessel's inequality for Fourier series.

Look it up.

6 Consider the Sturm-Liouville problem

$$u'' = \lambda u, \qquad u(0) = -u(\pi), \quad u'(0) = -u'(\pi).$$

(a) Prove that the problem is *symmetric* (or *self-adjoint* in the terminology of Folland), in the sense that

$$\langle u'', v\rangle = \langle u, v''\rangle,$$

for sufficiently differentiable functions u and v satisfying the boundary conditions.

(b) Find all solutions to the problem.

(a) By Lagrange's identity, it's enough to show that

$$[u'\bar{v} - u\bar{v}']_0^\pi = 0.$$

For each of the functions u', v, u, v', the value at π is -1 times the value at 0. Cancelling two minus signs from each term, we find that the values at the upper and lower end-point cancel.

(b) We distinguish between the cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.
When $\lambda > 0$, the solutions to the differential equation are $u(x) = A\cosh(\mu x) + B\sinh(\mu x)$, where $\mu = \sqrt{\lambda}$. The boundary conditions give after simplification

$$A(1 + \cosh(\mu\pi)) + B\sinh(\mu\pi) = 0, \qquad A\sinh(\mu\pi) + B(1 + \cosh(\mu\pi)) = 0.$$

This system has non-trivial solutions if and only if the determinant

$$(1 + \cosh(\mu\pi))^2 - \sinh(\mu\pi)^2 = 0,$$

that is, if

$$1 + \cosh(\mu\pi) = \pm\sinh(\mu\pi).$$

Since $1 + \cosh(\mu\pi)$ and $\sinh(\mu\pi)$ are both positive, the minus sign is impossible. Choosing the plus sign and expressing the hyperbolic functions in terms of exponential functions gives after simplification $1 + e^{-\mu\pi} = 0$, which is again impossible. Thus, there are no non-trivial solutions when $\lambda > 0$.
When $\lambda = 0$, we have $u(x) = A + Bx$. The boundary condition give $2A + \pi B = 2B = 0$, which implies $A = B = 0$. Again, there are no non-trivial solutions.
In the final case, when $\lambda < 0$, we have $u(x) = A\cos(\mu x) + B\sin(\mu x)$, where $\mu = \sqrt{-\lambda}$. Proceeding as before, we get a system with determinant

$$(1 + \cos(\mu\pi))^2 + \sin(\mu\pi)^2.$$

This vanishes only for $\cos(\mu\pi) = -1$ and $\sin(\mu\pi) = 0$, which is equivalent to μ being an odd integer (and positive since μ is a square root). Writing $\mu = 2k+1$, we find the solutions

$$u(x) = A\cos((2k + 1)x) + B\sin((2k + 1)x), \qquad k = 0, 1, 2, \ldots.$$

These are all the solutions to the given Sturm–Liouville problem.
Remark: In contrast to the problems that we have encountered during the course, the eigenspaces are two-dimensional rather than one-dimensional.