Supplementary Exercises on Solving General PDEs with Boundary Values (Chapter 4)

1. Consider the following inhomogeneous heat equation

$$\begin{cases} u'_t = 2u''_{xx} + F(x,t), & t > 0, \quad 0 < x < \pi, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < \pi. \end{cases}$$

(The extra term $F(x,t)$ is interpreted as time derivative of some external heat.)

(a) Let $F(x,t) = \sin x$ and $u_0(x) = \sin(3x)$. Solve the equation by using an Ansats to reduce it to a homogeneous equation.

(b) Let $F(x,t) = 1$ and $u_0(x) = 0$. Find a sine-series solution $\sum_n T_n(t) \sin nx$ by expanding $F(x)$ as sine-series $\sum_n c_n \sin nx$ and by solving the ODE[1] $T'_n(t) = -2n^2 T_n(t) + c_n$.

2. Solve the following inhomogeneous wave equation

$$\begin{cases} u''_{tt} = 2u''_{xx} + \sin x, & t > 0, \quad 0 < x < \pi, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = \sin 3x, & 0 < x < \pi, \\ u_t(x,0) = 0 & 0 < x < \pi. \end{cases}$$

Answer: 1a. $\frac{1}{2}\sin x - \frac{1}{2}e^{-2t}\sin x + e^{-18t}\sin 3x$.

1b. $-\frac{\pi}{2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3}(e^{-2(2k-1)^2 t} - 1)\sin(2k-1)x$.

2. $\frac{1}{2}\sin x + \sin(3x)\cos(\sqrt{2}t)$

[1] An ODE $T'(t) = aT(t) + b$ has a general solution of the form $\alpha e^{-at} + \beta$, α and β to be determined