Solution. Exam in MMG710/TMA362 Fourier Analysis, 2017-01-02.

1. Let f be 2π-periodic and $f(x) = x^2$, $-\pi < x < \pi$. The Fourier series of f is given by

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx. \tag{1}$$

(**a**) Evaluate the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

(**b**) Find the sum of the Fourier series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$, $0 < x < \pi$.

Solution The function f is continuous and $f'(x)$ is piece-wise continuous. Thus the Fourier series of f converges to f everywhere. Take $x = \pi$ and evaluate the Fourier series. Then $\cos n\pi = (-1)^n$ and we get $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Since $f'(x)$ is piece-wise continuous we can get its Fourier series by differentiating the Fourier series of $f(x)$:

$$f'(x) \overset{FourierSeries}{\sim} 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Change x to $x + \pi$. The Fourier series of $f'(x + \pi)$ is then

$$4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n(x + \pi) = 4\sum_{n=1}^{\infty} \frac{(-1)}{n} \sin nx$$

For $-\pi < x < \pi$, $f'(x) = 2x$. Now if $0 < x < \pi$ then $\pi < x + \pi < 2\pi$, $-\pi < x - \pi < 0$, and $f'(x + \pi) = f'(x - \pi) = 2(x - \pi)$, namely

$$2(x - \pi) \overset{FourierSeries}{\sim} 4\sum_{n=1}^{\infty} \frac{(-1)}{n} \sin nx, \quad 0 < x < \pi$$

Answer: $-\frac{1}{2}(x - \pi)$.

2. Let V be the three-dimensional subspace $V = \text{Span}\{\sin x, \cos^2 x, \sin^3 x\}$ of the Hilbert space $L^2(-\pi, \pi)$ and $f(x) = (\sin x) \cos 4x$.

(**a**) Find an orthonormal basis of V consisting of linear combination of trigonometric functions $\{\sin nx, \cos nx\}$.

(**b**) Find the orthogonal projection of f onto V and find the distance between f and the subspace V.

Solution We expand $\cos^2 x, \sin^3 x$ into Fourier series. We have

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x),$$

a linear combination of 1 and $\cos 2x$, and

$$\sin^3 x = \frac{1}{2}(1 - \cos 2x) \sin x = \frac{1}{2} \sin x - \frac{1}{4}(\sin 3x - \sin x) = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$$

a linear combination of $\sin x$ and $\sin 3x$. Thus

$$\{1 + \cos 2x, \sin x, \sin 3x\}$$

form an orthogonal basis of V and

$$\{\frac{1}{\sqrt{3\pi}}(1+\cos 2x), \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\sin 3x\}$$

form an orthonormal basis.

Now $f(x) = (\sin x)\cos 4x = \frac{1}{2}(\sin 5x - \sin 3x) = f_\perp(x) + f_V(x)$ with $f_\perp(x) = \frac{1}{2}\sin 5x$ being orthogonal to $1, \cos 2x, \sin x, \sin 3x$, with $f_V(x) = -\frac{1}{2}\sin 3x$ being in the space V. Thus the orthogonal projection $P_V f$ of f onto V is

$$P_V f = f_V, \quad f_V(x) = -\frac{1}{2}\sin 3x$$

and the distance between f and V is $\|f_\perp\| = \frac{1}{2}\pi$.

3. Let $f(x) = \frac{(\cos x)\sin 3x}{x}$. Find the Fourier transform $\widehat{f}(\xi)$ and compute the integrals $\int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)^2 dx$.

Solution Rewrite $(\cos x)\sin 3x = \frac{1}{2}(\sin 4x + \sin 2x)$ and $\frac{(\cos x)\sin 3x}{x} = \frac{1}{x}\sin 4x + \frac{1}{x}\sin 2x$, where the Fourier transform of each term is known:

$$\mathcal{F} : \frac{1}{x}\sin ax \to \pi\chi_a(\xi).$$

Thus

$$\mathcal{F} : \frac{(\cos x)\sin 3x}{x} \to \frac{\pi}{2}(\chi_4(\xi) + \chi_2(\xi)).$$

Now the two integrals can be computed using Fourier transform and Plancherel formula

$$\int_{-\infty}^{\infty} f(x)dx = \hat{f}(0) = \frac{\pi}{2}(1+1) = \pi$$

and

$$\int_{-\infty}^{\infty} f(x)^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)^2 d\xi = \frac{1}{2\pi}(\frac{\pi}{2})^2 \int_{-\infty}^{\infty} (\chi_4(\xi) + \chi_2(\xi))^2 d\xi.$$

We use the definition of χ_a to compute $(\chi_4(\xi) + \chi_2(\xi))^2$:

$$(\chi_4(\xi) + \chi_2(\xi))^2 = \chi_4(\xi)^2 + 2\chi_4(\xi)\chi_2(\xi) + \chi_2(\xi)^2 = \chi_4(\xi) + 2\chi_2(\xi) + \chi_2(\xi).$$

Thus

$$\int_{-\infty}^{\infty} f(x)^2 dx = 2\frac{\pi}{8}(4 + 2\cdot 2 + 2) = \frac{5\pi}{2}.$$

4. Solve the following homogeneous wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t > 0, \quad 0 < x < \pi \\ u_x(0,t) = 0,\ u_x(\pi,t) = 0, & t > 0 \\ u(x,0) = f(x),\ u_t(x,0) = 0 & 0 < x < \pi \end{cases}$$

where $f(x)$ is the function in the Problem 1. Find the position $u(x,t)$ of the wave at the time $t = \frac{\pi}{c}$. (Hint: Use the Fourier series (1)).

Solution The Hilbert space $L^2(0,\pi)$ has an orthogonal basis $\{\cos nx, n = 0, 1, \cdots\}$, each of the basis function satisfies $u_x(0) = 0,\ u_x(\pi) = 0$; the homogeneous equation $u_{tt} = c^2 u_{xx}$

with the boundary value has solutions $\cos nx\cos(nct), \cos nx\sin(nct)$. Thus the general solution is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \cos nx(a_n\cos(nct) + b_n\sin(nct)).$$

We evaluate the constants using the initial value: $b_n = 0$ since $u_t(x,0) = 0$ and a_n are the Fourier cosine-coefficient of $f(x)$. Thus

$$u(x,t) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx\cos(nct)$$

Now at the time $t = \frac{\pi}{c}$ we have $\cos(nct) = (-1)^n$ and

$$u(x,t) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$$

This is the function $f(x+\pi)$ in Problem 1. $f(x+\pi) = f(x-\pi) = (x-\pi)^2$, since $-\pi < x-\pi < 0$ for $0 < x < \pi$. Answer: At $t = \frac{\pi}{c}$, $u(x,t) = (x-\pi)^2$.

5. The Laplace tranform $F(z) = \mathcal{L}[f](z)$ of $f(t)$ is given by $F(z) = \frac{1}{z(z^2+a)}(1-e^{-z})$, where a is a constant, $a \in \mathbb{R}, a \neq 0$. Find the function $f(t)$. Determine the value of a so that the function $f(t)$ is bounded, i.e. $|f(t)| \leq M, t > 0$ for some constant M.

   Solution We write $a = -\lambda^2$ for $\lambda > 0$ or $\lambda = i\mu$, $\mu > 0$. Then $z^2 + a = (z-\lambda)(z+\lambda)$. We perform partial fraction to $\frac{1}{z(z^2+a)}$,

$$\frac{1}{z(z^2+a)} = \frac{1}{z(z-\lambda)(z+\lambda)} = \frac{1}{2\lambda^2}(-\frac{2}{z} + \frac{1}{z-\lambda} + \frac{1}{z+\lambda}).$$

   The Laplace inverse $f(t)$ of $F(z)$ is

$$f(t) = \frac{1}{2\lambda^2}(-2 + e^{\lambda t} + e^{-\lambda t})(H(t) - H(t-1))$$

   The function $f(t)$ is bounded if and only if $\lambda = i\mu$, that is $a = -\lambda^2 = \mu^2 > 0$.

6. Prove the Bessel's inequality for the Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$ of a function f in $L^2(-\pi,\pi)$.

   Solution See the textbook.