Mathematical Sciences (MV), Chalmers/GU
Monday, January 2, 2017, 8:30-12:30.

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Presence of teacher: $\sim 9:30$ and $\sim 11:30$

Tools: Only the attached sheets (2 pages) of formulas. No calculator or handbook is allowed.
(Language Dictionaries are allowed/Språlektion är tillåtet)

Exam in MMG710/TMA362 Fourier Analysis

1. Let f be 2π-periodic and $f(x) = x^2$, $-\pi < x < \pi$. The Fourier series of f is given by

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx. \tag{1}$$

(**a**) Evaluate the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

(**b**) Find the sum of the Fourier series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$, $\quad 0 < x < \pi$.

2. Let V be the three-dimensional subspace $V = \text{Span}\{\sin x, \cos^2 x, \sin^3 x\}$ of the Hilbert space $L^2(-\pi, \pi)$ and $f(x) = (\sin x) \cos 4x$.

(**a**) Find an orthonormal basis of V consisting of linear combinations of $\{\sin nx, \cos nx\}$.

(**b**) Find the orthogonal projection of f onto V and find the distance between f and the subspace V.

3. Let $f(x) = \frac{(\cos x)\sin 3x}{x}$. Find the Fourier transform $\widehat{f}(\xi)$ and compute the integrals $\int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)^2 dx$.

4. Solve the following homogeneous wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t > 0, \quad 0 < x < \pi \\ u_x(0,t) = 0,\ u_x(\pi,t) = 0, & t > 0 \\ u(x,0) = f(x),\ u_t(x,0) = 0 & 0 < x < \pi \end{cases}$$

where $f(x)$ is the function in the Problem 1. Find the position $u(x,t)$ of the wave at the time $t = \frac{\pi}{c}$. (Hint: Use the Fourier series (1)).

5. The Laplace transform $F(z) = \mathcal{L}[f](z)$ of $f(t)$ is given by $F(z) = \frac{1}{z(z^2+a)}(1 - e^{-2z})$, where a is a constant, $a \in \mathbb{R}, a \neq 0$. Find the function $f(t)$. Determine the value of a so that the function $f(t)$ is bounded, i.e. $|f(t)| \leq M, t > 0$ for some constant M.

6. Prove the Bessel's inequality for the Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$ of a function f in $L^2(-\pi, \pi)$.

Grades: 6 problems each of 4 points.
MMG710: G (12-17 p.), VG (18-24 p.). **TMA362**: 3 (12-14 p.), 4 (15-17 p.), 5 (18-24 p.)

Basic Fourier Transforms and Laplace Transforms

0.1 Fourier transform $\mathcal{F} : f(x) \mapsto \widehat{f}(\xi)$. ($a > 0$ and $c \in \mathbb{R}$ are constants)

$f(x-c)$	$e^{-ic\xi}\widehat{f}(\xi)$
$e^{i\alpha x}f(x)$	$\widehat{f}(\xi-\alpha)$
$f(ax)$	$a^{-1}\widehat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\widehat{f}(\xi)$
$xf(x)$	$i(\widehat{f})'(\xi)$
$(f*g)(x)$	$\widehat{f}(\xi)\widehat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\widehat{f}*\widehat{g})(\xi)$
$e^{-a\frac{x^2}{2}}$	$\sqrt{\frac{2\pi}{a}}\, e^{-\frac{\xi^2}{2a}}$
$(x^2+a^2)^{-1}$	$\frac{\pi}{a}e^{-a\|\xi\|}$
$e^{-a\|x\|}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x)$	$2\xi^{-1}\sin a\xi$
$x^{-1}\sin ax$	$\pi\chi_a(\xi)$

0.2 Laplace transforms $\mathcal{L} : f(t) \mapsto F(z) = \mathcal{L}f(z)$. ($a > 0$ and $c \in \mathbb{C}$ are constants)

$H(t-a)f(t-a)$			$e^{-az}F(z)$		
$e^{ct}f(t)$			$F(z-c)$		
$f(at)$			$a^{-1}F(a^{-1}z)$		
$f'(t)$			$zF(z)-f(0)$		
$f''(t)$			$z^2F(z)-zf(0)-f'(0)$		
$f*g$			FG		
$H*f(t)=\int_0^t f(s)ds$			$z^{-1}F(z)$		
$tf(t)$			$-F'(z)$		
t^ne^{ct}			$\frac{n!}{(z-c)^{n+1}}$		
$\sin ct$	resp.	$\cos ct$	$\frac{c}{z^2+c^2}$	resp.	$\frac{z}{z^2+c^2}$
$\sinh ct$	resp.	$\cosh ct$	$\frac{c}{z^2-c^2}$	resp.	$\frac{z}{z^2-c^2}$

Some formulas in Fourier analysis

Trigonometric identities

$$e^{ix} = \cos x + i \sin x, \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y, \qquad \sin(x+y) = \sin x \cos y + \cos x \sin y,$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2},$$

$$\sin x \, \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}, \qquad \cos x \, \cos y = \frac{\cos(x-y) + \cos(x+y)}{2},$$

$$\sin x \, \cos y = \frac{\sin(x+y) + \sin(x-y)}{2}.$$

Hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Characteristic and Heaviside functions

$$\chi_{[a,b]}(x) = \begin{cases} 1, & a < x < b, \\ 0, & \text{otherwise.} \end{cases} \qquad \chi_a(x) := \chi_{[-a,a]}(x).$$

$$H(t) = \chi_{[0,\infty)}(t) = \begin{cases} 1, & t > 0, \\ 0, & \text{else.} \end{cases}$$

Gamma function $\Gamma(a)$

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx, \qquad a > 0$$

$$\Gamma(a+1) = a\Gamma(a), \qquad \Gamma(n+1) = n!$$