Mathematical Sciences (MV), Chalmers/GU
Monday, October 27, 2014, 8:30-12:30.

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Presence of teacher: $\sim 9:30$ and ~ 11.30

Tools: Only the attached sheets (2 pages) of formulas. No calculator or handbook is allowed.
(Language Dictionaries are allowed/Språklektion är tillåtet)

Exam in MMG710/TMA362 Fourier Analysis

1. In which space C^k (i.e. functions with continuous derivatives $f^{(j)}(\theta)$, $j \le k$, for all $\theta \in \mathbb{R}$) are the following periodic functions? Find the best (i.e. the largest) k.

$$(\mathbf{a}) \quad \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^2}, \ (\mathbf{b}) \quad \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{5^n}, \ (\mathbf{c}) \quad \sum_{n=1}^{\infty} \frac{\sin(2^n\theta)}{5^n}.$$

   Motivate your answers. (1+2+1 p)

2. Compute the following integral

$$\int_{-\infty}^{\infty} \frac{\sin(x)\cos(2x)}{x(x^2+1)} dx$$

3. Solve the following ordinary differential equation

$$u''(t) - 4u(t) = f(t), \ u(0) = 0, \ u'(0) = 1,$$

   where

$$f(t) = 8H(t-1) = \begin{cases} 8, & t \ge 1 \\ 0, & \text{else} \end{cases}$$

4. Solve the following inhomogeneous wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} + t\sin(2x), & t > 0, \quad 0 < x < \pi \\ u(0,t) = 0, \ u(\pi,t) = 0, & t > 0 \\ u(x,0) = x(\pi - x), & 0 < x < \pi. \end{cases}$$

   (The initial velocity $u_t(x,0) = g(x), 0 < x < \pi$, is an arbitrary function.) You may use (without proof) the following Fourier sine series on $(-\pi, \pi)$

$$x(\pi - |x|) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}.$$

5. Evaluate the sum of the following series by using the above Fourier expansion. Motivate your answer.

$$(\mathbf{a}) \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}, \quad (\mathbf{b}) \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}.$$

6. Formulate and prove *the Theorem on Uniform Convergence for Fourier Series* of 2π-periodic C^1-functions.

Grades: 6 problems each of 4 points.
MMG710: G (12-17 p.), VG (18-24 p.). TMA362: 3 (12-14 p.), 4 (15-17 p.), 5 (18-24 p.)

Basic Fourier Transforms and Laplace Transforms

0.1 Fourier transform $\mathcal{F} : f(x) \mapsto \widehat{f}(\xi)$. ($a > 0$ and $c \in \mathbb{R}$ are constants)

$f(x-c)$	$e^{-ic\xi}\widehat{f}(\xi)$
$e^{i\alpha x}f(x)$	$\widehat{f}(\xi-\alpha)$
$f(ax)$	$a^{-1}\widehat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\widehat{f}(\xi)$
$xf(x)$	$i(\widehat{f})'(\xi)$
$(f*g)(x)$	$\widehat{f}(\xi)\widehat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\widehat{f}*\widehat{g})(\xi)$
$e^{-a\frac{x^2}{2}}$	$\sqrt{\frac{2\pi}{a}}\,e^{-\frac{\xi^2}{2a}}$
$(x^2+a^2)^{-1}$	$\frac{\pi}{a}e^{-a\lvert\xi\rvert}$
$e^{-a\lvert x\rvert}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x)$	$2\xi^{-1}\sin a\xi$
$x^{-1}\sin ax$	$\pi\chi_a(\xi)$

0.2 Laplace transforms $\mathcal{L} : f(t) \mapsto F(z) = \mathcal{L}f(z)$. ($a > 0$ and $c \in \mathbb{C}$ are constants)

$H(t-a)f(t-a)$	$e^{-az}F(z)$
$e^{ct}f(t)$	$F(z-c)$
$f(at)$	$a^{-1}F(a^{-1}z)$
$f'(t)$	$zF(z)-f(0)$
$f''(t)$	$z^2F(z)-zf(0)-f'(0)$
$f*g$	FG
$H*f(t)=\int_0^t f(s)ds$	$z^{-1}F(z)$
$tf(t)$	$-F'(z)$
$t^n e^{ct}$	$\frac{n!}{(z-c)^{n+1}}$
$\sin ct$ resp. $\cos ct$	$\frac{c}{z^2+c^2}$ resp. $\frac{z}{z^2+c^2}$
$\sinh ct$ resp. $\cosh ct$	$\frac{c}{z^2-c^2}$ resp. $\frac{z}{z^2-c^2}$

Some formulas in Fourier analysis

Trigonometric identities

$$e^{ix} = \cos x + i \sin x, \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y, \qquad \sin(x+y) = \sin x \cos y + \cos x \sin y,$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2},$$

$$\sin x \ \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}, \qquad \cos x \ \cos y = \frac{\cos(x-y) + \cos(x+y)}{2},$$

$$\sin x \ \cos y = \frac{\sin(x+y) + \sin(x-y)}{2}.$$

Hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Characteristic and Heaviside functions

$$\chi_a(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| < a. \end{cases}$$

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & \text{else}. \end{cases}$$