Mathematical Sciences (MV), Chalmers/GU
Monday, August 17, 2015, 8:30-12:30.
Phone: 0703-088304 Gustav Kettil
Presence of teacher: $\sim 9:30$ and $\sim 11:30$

Tools: Only the attached sheets (2 pages) of formulas. No calculator or handbook is allowed.
(Language Dictionaries are allowed/Språklektion är tillåtet)

Exam in MMG710/TMA362 Fourier Analysis

1. Let $f(x)$ be the 2π-periodic function on $\mathbb{R}$ such that $f(x) = |x|, -\pi < x \le \pi$. The Fourier expansion of $f(x)$ is
$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}. \tag{1}$$
Find the Fourier series expansions of the periodic functions $f'(x)$ and $\int_{-\pi}^{x}(f(t) - \frac{\pi}{2})dt$.

2. Consider the space $L^2(0,1)$ with the norm square $\|f\|^2 = \int_0^1 f(x)^2 dx$ and its subspace $\mathcal{V}$ consisting of functions of the form $a + bx$, a, b being constants. Find the distance between the function $f(x) = x^2$ and the space $\mathcal{V}$. (Recall that the distance between f and the space $\mathcal{V}$ is defined to be the minimum among all $\|f - v\|$, $v \in \mathcal{V}$.)

3. The Fourier transform $\widehat{f}(\xi)$ of a smooth function f on $\mathbb{R}$ is given by
$$\widehat{f}(\xi) = 0,\ \xi \notin [0,1]; \quad \widehat{f}(\xi) = \frac{1}{1+\xi},\ \xi \in [0,1].$$
(**a**) Find the values $f(0)$ and $f'(0)$. (**b**) Compute the integral $\int_{-\infty}^{\infty}|f(x)|^2 dx$.

4. Solve the following wave equation
$$\begin{cases} u_{tt} = c^2 u_{xx} + \cos x, & t > 0, \quad 0 < x < \pi \\ u_x(0,t) = 0,\ u_x(\pi,t) = 0, & t > 0 \\ u(x,0) = |x|,\ u_t(x,0) = 0 & 0 < x < \pi. \end{cases}$$
(Hint: Use the Fourier series of $|x|$ in Problem 1 above.)

5. (a) Give an example of a piece-wise continuous 2π-periodic function whose Fourier series is not absolutely convergent.

   (b) Prove the Lagrange identity for the Sturm-Liouville operator $L(f) = f''$ on the interval $[a,b]$, namely, find a formula of the form $\int_a^b L(f)g dx = \int_a^b fL(g)dx + R$ and compute the reminder term R.

6. Formulate and prove a statement on *uniform convergence of Fourier series*.

Grades: 6 problems each of 4 points.
MMG710: G (12-17 p.), VG (18-24 p.). **TMA362**: 3 (12-14 p.), 4 (15-17 p.), 5 (18-24 p.)

Basic Fourier Transforms and Laplace Transforms

0.1 Fourier transform $\mathcal{F} : f(x) \mapsto \widehat{f}(\xi)$. ($a > 0$ and $c \in \mathbb{R}$ are constants)

$f(x-c)$	$e^{-ic\xi}\widehat{f}(\xi)$
$e^{i\alpha x}f(x)$	$\widehat{f}(\xi-\alpha)$
$f(ax)$	$a^{-1}\widehat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\widehat{f}(\xi)$
$xf(x)$	$i(\widehat{f})'(\xi)$
$(f*g)(x)$	$\widehat{f}(\xi)\widehat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\widehat{f}*\widehat{g})(\xi)$
$e^{-a\frac{x^2}{2}}$	$\sqrt{\frac{2\pi}{a}}\,e^{-\frac{\xi^2}{2a}}$
$(x^2+a^2)^{-1}$	$\frac{\pi}{a}e^{-a\lvert\xi\rvert}$
$e^{-a\lvert x\rvert}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x)$	$2\xi^{-1}\sin a\xi$
$x^{-1}\sin ax$	$\pi\chi_a(\xi)$

0.2 Laplace transforms $\mathcal{L} : f(t) \mapsto F(z) = \mathcal{L}f(z)$. ($a > 0$ and $c \in \mathbb{C}$ are constants)

$H(t-a)f(t-a)$	$e^{-az}F(z)$
$e^{ct}f(t)$	$F(z-c)$
$f(at)$	$a^{-1}F(a^{-1}z)$
$f'(t)$	$zF(z)-f(0)$
$f''(t)$	$z^2F(z)-zf(0)-f'(0)$
$f*g$	FG
$H*f(t)=\int_0^t f(s)ds$	$z^{-1}F(z)$
$tf(t)$	$-F'(z)$
$t^n e^{ct}$	$\frac{n!}{(z-c)^{n+1}}$
$\sin ct$ resp. $\cos ct$	$\frac{c}{z^2+c^2}$ resp. $\frac{z}{z^2+c^2}$
$\sinh ct$ resp. $\cosh ct$	$\frac{c}{z^2-c^2}$ resp. $\frac{z}{z^2-c^2}$

Some formulas in Fourier analysis

Trigonometric identities

$$e^{ix} = \cos x + i\sin x, \qquad \cos x = \frac{e^{ix}+e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix}-e^{-ix}}{2i},$$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y, \qquad \sin(x+y) = \sin x\cos y + \cos x\sin y,$$
$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2},$$
$$\sin x\ \sin y = \frac{\cos(x-y)-\cos(x+y)}{2}, \qquad \cos x\ \cos y = \frac{\cos(x-y)+\cos(x+y)}{2},$$
$$\sin x\ \cos y = \frac{\sin(x+y)+\sin(x-y)}{2}.$$

Hyperbolic functions

$$\cosh x = \frac{e^{x}+e^{-x}}{2}, \qquad \sinh x = \frac{e^{x}-e^{-x}}{2}.$$

Characteristic and Heaviside functions

$$\chi_a(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| < a. \end{cases}$$

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & \text{else.} \end{cases}$$

Gamma function $\Gamma(a)$

$$\Gamma(a) = \int_0^\infty e^{-x}x^{a-1}dx, \qquad a > 0$$
$$\Gamma(a+1) = a\Gamma(a), \qquad \Gamma(n+1) = n!$$